Is Faraday's Disk Dynamo a Flux-Rule Exception?

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A very condensed version of this long note is at [567].

1 Problem

In sec. 17.1 of [369], Feynman noted that the differential form of "Faraday's law" is,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

and then argued that for a fixed loop one can deduce the integral form of this "law" as,

$$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\text{surface of loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = -\frac{d}{dt}(\text{magnetic flux through loop}), \tag{2}$$

which is often called the "flux rule". In sec. 17.2, he considered an experiment of Faraday from 1831,[1] sketched below, and claimed that this is an example of an *exception to the "flux rule"*, where one should instead consider the motional $\mathcal{EMF} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$.
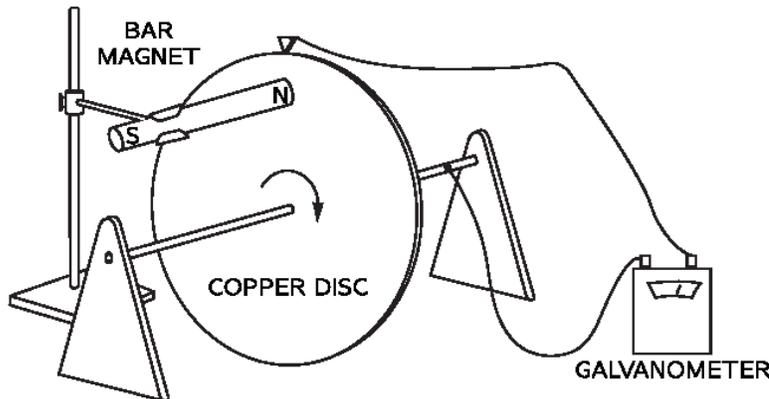

Fig. 17–2. When the disc rotates there is an emf from $\boldsymbol{v} \times \boldsymbol{B}$, but with no change in the linked flux.

However, a recent paper [552] expressed the view that Feynman and many others are "confused" about this example, and that it is well explained by the "correct" interpretation of the "flux rule".

What's going on here?

2 Solution

In my view, the issue is that examples of magnetic induction can be analyzed more than one way, and whenever there is more than one way of doing anything, some people become overly enthusiastic for their preferred method, and imply that other methods are incorrect.

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[1]See Art. 99 of [298]. Faraday's dynamo is the inverse of a motor demonstrated by Barlow in 1822 [50], and discussed by Sturgeon in 1826 [75].

As reviewed in Appendix A.17 below, Faraday discovered electromagnetic induction in 1831 [81], and gave an early version of what is now called Faraday's law in Art. 256 of [82]: *If a terminated wire move so as to cut a magnetic curve, a power is called into action which tends to urge an electric current through it.* One of his last statements of this law was given in Art. 3115 of [137] (1851): *The quantity of electricity thrown into a current is directly as the amount of* (magnetic) *curves intersected.*

The interpretation of Faraday's views on magnetic induction in a closed loop,[2] whose shape may or may not vary with time in a magnetic field that may or may not vary with time, as,

$$\mathcal{EMF} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\frac{d}{dt}\int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area}, \tag{3}$$

originated with Maxwell.[3,4] However, this relation is often tricky to apply (due to ambiguities as to where the loop is, and to the meaning of $\mathcal{EMF}$ and of d/dt), so many people, including Feynman, recommend splitting the calculation into two pieces,[5,6]

$$\mathcal{EMF} = \mathcal{EMF}_{\text{fixed loop}} + \mathcal{EMF}_{\text{motional}}, \tag{4}$$

where,[7]

$$\mathcal{EMF}_{\text{fixed loop}} = -\frac{\partial}{\partial t}\int_{\text{loop at time } t} \mathbf{B} \cdot d\mathbf{Area} = -\int_{\text{loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l}, \tag{5}$$

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[2]The term "loop" or "circuit" has the implication to "mathematicians" of being closed, so the adjective "closed" in often omitted in discussions of eq. (3). Furthermore, in the "mathematical" sense, a loop or circuit need not be associated with matter, *i.e.*, electrical conductors.

However, eq. (3) does not apply to a line that connects two different points, which is often called an "open circuit".

[3]The earliest statement of this law by Maxwell was in a letter to W. Thomson, Nov. 13, 1854, p. 703 of [307]: *The electromotive force along any line is measured by the number of lines of pol$^{\text{m}}$ (i.e., of the magnetic field) wh(ich) that line cuts in unit of time. Hence the electromotive force round a given circuit depends on the decrease of the number of lines wh: pass thro it in unit of time, that is, on the decrease of the whole pol$^{\text{m}}$ of any surface bounded by the circuit.*

[4]In Art. 530 of his *Treatise* [248], Maxwell considered electromagnetic induction in four different configurations, and then stated in Art. 531:
The whole of these phenomena may be summed up in one law. When the number of lines of magnetic induction which pass through the secondary circuit in the positive direction is altered, an electromotive force acts round the circuit, which is measured by the rate of decrease of the magnetic induction through the circuit.

[5]As will be reviewed the historical Appendix below, Faraday's initial discovery of electromagnetic induction was between two fixed loops, but almost all of his subsequent effort involved moving conductors where the induced $\mathcal{EMF}$ is motional. As such, the first efforts at a theory of electromagnetic induction, by Neumann [116] and Weber [121], emphasized only the motional $\mathcal{EMF}$, and arrived at equivalents of eq. (6), but without use of the concept of the magnetic field.

[6]The partition (4) is implicit in Art. 598 of Maxwell's *Treatise* [191], as reviewed in Appendix A.28.4.7 below, but may have been first given explicitly by Heaviside (1885) [220]. It was advocated by Steinmetz (1908), p. 1352 of [272], where he indicated that the "flux rule" should be applied only to fixed loops, and that the Biot-Savart/Lorentz force law should be considered in cases of moving circuits/conductors.

[7]Our $\mathcal{EMF}_{\text{fixed loop}}$ is called "transformer" $\mathcal{EMF}$ in some engineering discussions.

using eq. (1) and Stoke's theorem, and,

$$\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}, \tag{6}$$

in which $\mathbf{v}$ is the velocity (in the inertial lab frame of the calculation) of an element $d\mathbf{l}$ of the loop (which is a line, but which may or may not be inside a conductor).

Maxwell showed (although apparently not very clearly) in Arts. 598-599 of [248] that eq. (3) can also be written as,[8]

$$\mathcal{EMF} = \oint \left(\mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} \right) \cdot d\mathbf{l}, \tag{7}$$

where the electromagnetic fields $\mathbf{B}$ and $\mathbf{E}$ can be related to a vector potential $\mathbf{A}$ and a scalar potential V (called Ψ by Maxwell in Arts. 598-601 of [248]) according to,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad \mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}, \tag{8}$$

such that,[9]

$$\mathcal{EMF} = \oint (\mathbf{v} \times \mathbf{B} + \mathbf{E}) \cdot d\mathbf{l} = \mathcal{EMF}_{\text{motional}} + \mathcal{EMF}_{\text{fixed loop}}, \tag{9}$$

since for a closed, fixed loop, $\oint \boldsymbol{\nabla}V \cdot d\mathbf{l} = 0$ and $\oint \mathbf{E} \cdot d\mathbf{l} = - \oint \partial \mathbf{A}/\partial t \cdot d\mathbf{l}$.

Thus, the two methods, eq. (3) and eq. (9), of computing induced $\mathcal{EMF}$s, give the same results (when correctly computed). As illustrated in sec. 2.4 below, in examples with moving circuit elements, the method of eq. (9) is generally easier to apply.[10]

2.1 What Does $\mathcal{EMF}$ Mean?

A lingering issue is that the symbol $\mathcal{EMF}$ has not been defined independent of eqs. (3) and (5)-(6).

It is generally agreed that $\mathcal{EMF}$ means electromotive force, but what does the latter mean?[11]

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[8]The equivalence of eqs. (3) and (7) and (9) assumes that any motion of the circuit is continuous. If "switches" are involved, it appears better to use eq. (9) rather than eq. (3), as illustrated in sec. 2.4.3 below.

[9]The first clear statement of the equivalence of eqs. (3) and (9) may be in sec. 86 of the text of Abraham (1904) [268], which credits Hertz (1890) [242] for inspiration on this. As discussed in Appendix A.29 below, Boltzmann understood this equivalence in 1891, but did not express it very clearly. An early verbal statement of this in the American literature was by Steinmetz (1908), pp. 1352-53 of [272], with a more mathematical version given by Bewley (1929) in Appendix I of [294]. Textbook discussions in English include that by Becker, pp. 139-142 of [297], by Sommerfeld, pp. 286-288 of [345], by Panofsky and Phillips, pp. 160-163 of [360], and by Zangwill, sec. 14.4 of [525].

[10]In mechanics, torque analyses can be performed about any point, but in examples without fixed axles, it is generally easiest to use the center of mass of the system as the reference point [555].

[11]It is claimed on p. 3 of [58] (1823, perhaps the first textbook on electromagnetism), and also in [412], that Volta [20, 21] coined this term. Presumably, this is a translation of Volta's term *force motrice électrique* on p. 254 of [21] (1801).

Fechner (1831), p. 225 of [80], may have been the first to interpret the (scalar) symbol $\mathcal{E}$ in Ohm's law [76],

$$\mathcal{E} = IR, \tag{10}$$

as the *electromotorische Kraft*, *i.e.*, electromotive force (although this $\mathcal{E}$ has dimensions of energy, not force).[12] Ohm and Fechner studied only steady currents, in which case the electromotive "force" between two points a and b is equal to the work done by electric effects when moving a unit electric charge from between two points a and b in a static electric field,[13,14]

$$\mathcal{E}(a,b) = V(a) - V(b) = \int_a^b -\boldsymbol{\nabla} V \cdot d\mathbf{l} = \int_a^b \mathbf{E} \cdot d\mathbf{l}. \tag{12}$$

Although Faraday's first report (1831) [81] of electromagnetic induction was based on time-dependent magnetic flux in a system of two coils at rest in the lab, all of his subsequent studies involved moving elements. This led Neumann, in sec. 1 of his great paper of 1845 [116] which introduced the concept of inductance, to discuss the (vector) *elektromotorische Kraft* due to current I' in segment $d\mathbf{l}'$ on a segment $d\mathbf{l}$ of a conductor that carries electric current I,[15]

$$d^2\mathbf{F}_{elektromotorishche} = I\, d\mathbf{l} \times d\mathbf{B} = \mu_0 I I'\, d\mathbf{l} \times \frac{d\mathbf{l}' \times \mathbf{r}}{r^3}. \tag{13}$$

Since Neumann (more clearly than Faraday) had a vision of electric current as due to electric charge in motion, Neumann's (vector) *elektromotorische Kraft* corresponds to the (vector) integrand of the (scalar) motional $\mathcal{EMF}$ (6).

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[12]The formulation of Ohm's law as eq. (10) suggests the interpretation that "electromotive force" is the cause of electric currents. This view was endorsed by Maxwell in Art. 569 of [248].

[13]The was first pointed out by Kirchhoff in [129], and mentioned by Maxwell in Art. 69 of [247].

[14]One can consider a "field theory of batteries," in which the $\mathcal{EMF}$ of a battery is associated with a nonelectrostatic field $\mathbf{E}'$, that is nonzero only inside the battery. Then, the current density $\mathbf{J}$ inside a material (at rest) of conductivity σ, can be written as $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}')$. If the battery is, say, a cylinder of length L_B and cross sectional area A_B, and is not connected to anything, $\mathbf{J}_B = 0$, such that $\mathbf{E}'_B = -\mathbf{E}_B$, where the subscript B means inside the battery. Approximating the fields inside the battery as uniform, the magnitudes of the fields inside it are $E'_B = E_B = \mathcal{E}/L_B$, where $\mathcal{E}$ is the $\mathcal{EMF}$ of the battery.

If the battery is connected to an external resistance R_R, where the subscript R refers to the external resistor, then current $I = \mathcal{E}/(R_R + R_B)$ flows in the circuit, where $R_B = L_B/\sigma_B A_B$ is the internal resistance of the battery. The $\mathcal{EMF}$ between the terminals of the battery, along the external wires/resistor has magnitude $\mathcal{E}_R = IR_R = \mathcal{E}R_R/(R_R + R_B)$. Since the electrostatic field $\mathbf{E}$ obeys $\oint_{\mathrm{circuit}} \mathbf{E} \cdot d\mathbf{l} = 0$, we infer that the electric field inside the battery now is $E_B = -\mathcal{E}_R/L_B = -\mathcal{E}R_R/L_B(R_R + R_B)$ (and points from the $+$ to the $-$ terminal of the battery). In addition, the current density inside the battery is related by $J_B = I/A_B = \sigma_B(E_B + E'_B)$, such that

$$E'_B = \frac{I}{\sigma_B A_B} - E_B = \frac{\mathcal{E}}{R_R + R_B}\frac{R_B}{L_B} + \frac{\mathcal{E}R_R}{L_B(R_R + R_B)} = \frac{\mathcal{E}}{L_B}, \tag{11}$$

as was also the case when the battery was not connected to the resistor.

The "battery field theory" was introduced by Abraham (1904), Secs. 50-52 of [268]. See also [297, 365, 412].

[15]Neumann had no concept of the magnetic field $\mathbf{B}$, or of vector analysis, and only wrote a scalar equivalent of the second form of eq. (13).

Thus, early in the history of electromagnetic induction, two meanings of $\mathcal{EMF}$, eqs. (5) and (6), became prominent, and this tradition survives to the present day, as exemplified by Feynman's discussion [369].

Despite Maxwell's attempt in Arts. 598-599 of [248] to merge the two concepts of $\mathcal{EMF}$ into one,[16] only after Lorentz' clarification (1895), eq. (V), p. 21 of [257], that the electromagnetic force on charge q is,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),\tag{14}$$

did some people begin to accept that $\mathcal{EMF}$ could/should be given by eq. (9),[17] where the $\mathcal{EMF}$ is the work done according to the force law (14) on unit charge as it traverses a circuit.[18]

Even with this clarification, analysis of magnetic induction via eqs. (4)-(6) remains more appealing to many people than use of eq. (3), whose equivalence to eq. (9) is not always evident. For many, the notion of motional $\mathcal{EMF}$ as eq. (6) gives a better physical understanding of examples like Faraday's disk dynamo than does consideration of magnetic flux through a deforming circuit.

2.1.1 Faraday's Local (Electromotive) Force

Faraday was a great advocate of local, rather than action-at-a-distance, explanations of physical phenomena. As such, he gave little consideration to more global considerations like the analysis of an entire electric circuit.

In 1831, Art. 99 of [81], described the action of his disk dynamo in a manner readily interpreted (as later done by Neumann) as an induced current element being due to the force element (13).[19]

Perhaps the closest Faraday came to a statement of "his" flux law (3) was in Art. 3115 of [137] (1851): *The quantity of electricity thrown into a current is directly as the amount of* (magnetic) *curves intersected.*[20]

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[16]Maxwell used the term *electromotive force* to mean both the force per unit charge, $\mathbf{v} \times \mathbf{B} + \mathbf{E}$ in case of a moving charge, eq. (10) of Art. 599, and also the line integral of this quantity (called the $\mathcal{EMF}$ in the present note). As he stated at the end of Art. 598: *The electromotive force at a point, or on a particle, must be carefully distinguished from the electromotive force along an arc or a curve, the latter quantity being the line integral of the former.*

Despite this encouragement to the reader, Maxwell did not give the integral form of our eq. (9) in Arts. 598-599, but only the integrand. As such, its physical significance was missed by most readers. For additional discussion, see Appendix A.28.4.6 below.

[17]Lorentz (1892), p. 405 of [246], still identified the last form of eq. (5) as the *force électromotrice*, and wrote the "flux law" in eq. (42), p. 416, as $\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -(d/dt) \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area}$. And, in 1903, eq. 27, p. 83 of [267], Lorentz again referred to this equation, but with the proviso that the loop be at rest.

The equivalence of eqs. (3) and (9) was discussed in the influential (German) text of Abraham (1904) [268], sec. 86, p. 398. But, his analysis of unipolar induction, sec. 88, p. 409, was based only on motional $\mathcal{EMF}$.

[18]However, this definition remains in some conflict with circuit analysis, where it is assumed that a unique scalar $\mathcal{EMF}$/voltage can be assigned at any junction between circuit elements, as discussed in sec. 3.2 below.

[19]See Appendix A.17.4 below.

[20]See also Appendix A.17.12 below.

As can be seen in some of the examples in sec. 2.4 below, Faraday's notion of $\mathcal{EMF}$ as due to the local "cutting" of magnetic field lines by a conductor is sometimes more compelling than the greater abstraction (3) of changes in the global magnetic flux linked by a circuit.

2.2 When Can an $\mathcal{EMF}$ Be Measured?

A disconcerting aspect of the concept of $\mathcal{EMF}$ is that although it is defined for imaginary paths/circuits, it is not measurable in practice unless a conductor exists along the path that defines the $\mathcal{EMF}$.

When a conductor lies along the path, the $\mathcal{EMF}$ between two points on it can (generally) be measured with a voltmeter (aka galvanometer), while the voltmeter reads "nothing" when not sampling a test conductor.[21] As such, an $\mathcal{EMF}$ is a less physical concept than electric and magnetic fields, although these are already somewhat abstract.

The greater abstraction of an $\mathcal{EMF}$ may contribute to the ongoing differences of opinion as how best to think about it.

2.3 Carter's Rule

An interesting qualification to the generalized flux law (3) was given by Carter (1967) on p. 170 of his "engineering" textbook [376]:[22]

The equation $\mathcal{E} = -d\Phi/dt$ always gives the induced e.m.f. correctly, provided the flux-linkage is evaluated for a circuit so chosen that at no point are particles of the material moving across it.[23]

I believe this statement should also include the proviso *..., and at no time is there a discontinuous change in the linked flux.*

Note that the mention of *the material* implies the concept of $\mathcal{EMF}$ was only considered by Carter for conducting circuits, and not for "imaginary" closed curves. We comment further in sec. 3.1 below on this type of distinction in the use of terms by "engineers" and "physicists".

We illustrate applications of eqs. (3) and (9), as well as Carter's rule, in seven examples in the next section.

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[21]Strictly, a voltmeter that is not connected to a test conductor is a kind of antenna, and can give a nonzero AC-voltage reading when electromagnetic radiation is present. In such cases, it is difficult to interpret the reading of the voltmeter as an $\mathcal{EMF}$ [503].

[22]An earlier attempt at clarification of the generalized flux law was given by Bewley [370], where he spoke of the *substitution of circuits*, in which cases no $\mathcal{EMF}$ is generated. The present author finds Bewley's argument difficult to apply.

[23]That is, a valid path through the interior of a material must be at rest with respect to that material.

2.4 Seven Examples

2.4.1 Faraday's Disk Dynamo

We illustrate different analyses of Faraday's disk dynamo in case of a spatially uniform magnetic field, using the figures on the next page (from [324] and [326]).[24]

Field Constant in Time

We start by considering the fixed loop ABCDOA in the figure on the right below.

There is no magnetic flux through this loop, so $\mathcal{EMF}_{\text{fixed loop}} = 0$ here. On the other hand, segment DO, of length a, rotates with angular velocity ω, so a point on this segment at distance r from the axis has velocity $v = \omega r$ perpendicular to $\mathbf{B}$, such that,

$$\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \int_a^0 \omega r B \, dr = -\frac{a^2 B \omega}{2} \ . \tag{15}$$

Note that if we instead considered the loop ABCDD′OA, we again have $\mathcal{EMF}_{\text{fixed loop}} = 0$, while now segment D′O rotates, and again the motional (and total) $\mathcal{EMF}$ is given by eq. (15).
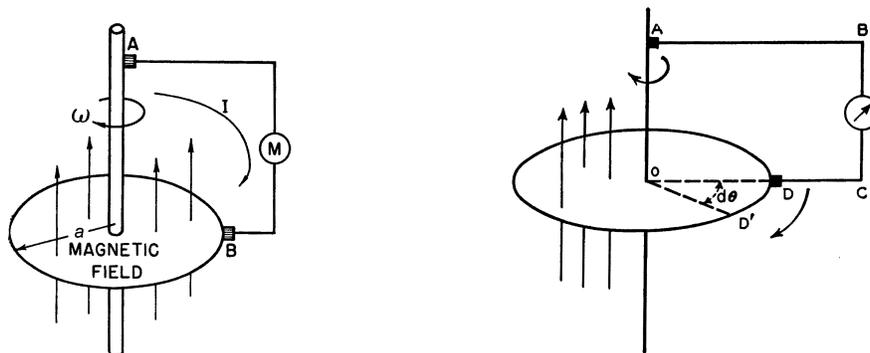
To use the generalized "flux law" (3) for loop ABCDOA, we compare the flux through the loop at times t and $t + dt$. At the latter time, segment DO has rotated by angle $d\theta = \omega \, dt$, and we must suppose that a new segment DD′ is added to the loop so that it remains closed, since eq. (3) only applies to closed loops. Then, the magnetic flux through loop ABCDD′OA is $\Phi_{\mathbf{B}}(t + dt) = B \, a^2 \, d\theta/2 = a^2 B \omega \, dt/2$, and total $\mathcal{EMF}$ is,[25]

$$\mathcal{EMF}_{\text{total}} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\frac{a^2 B \omega}{2} \ . \tag{16}$$

"Seat" of the $\mathcal{EMF}$

An issue for many people concerning Faraday's disk dynamo is the location of the "seat" of the $\mathcal{EMF}$ (16). In this context, the notion of the motional $\mathcal{EMF}$ is appealing in suggesting that the Lorentz force $e\mathbf{v} \times \mathbf{B}$ on moving charges in the disk identifies the "seat" of the $\mathcal{EMF}$ as being in the copper disk, while this is left ambiguous by the more abstract relation (3).

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[24]The earliest use in English of the generalized flux rule (3) for Faraday's disk dynamo via a deforming circuit that I have found is in [326] (1949), although it seems that Poincaré considered this in 1900 [264].

[25]According to Carter's Rule of sec. 2.3, loop ABCDD′OA rather than ABCDOA must be used with eq. (3).

"Drum" Dynamo
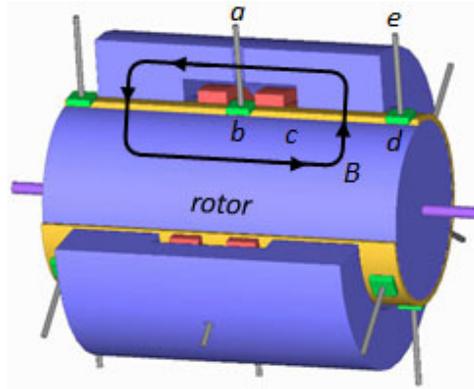
Practical homopolar generators are more often constructed with a "drum" geometry, as sketched above (from [524]).

Here, a copper-coated iron cylinder rotates inside a cylindrical electromagnet, and a motional $\mathcal{EMF}$ vBl_{cd}, where v is the azimuthal velocity of the copper cylinder, is generated along the rotating copper between sliding contacts at c and d (and elsewhere). One can also consider the circuit $abcdea$ as having a moving segment bc, and use the generalized flux law (3) to compute the $\mathcal{EMF}$. But again, the computation of the motional $\mathcal{EMF}$ is more appealing to many (particularly in the "engineering" community).

Field Sinusoidal in Time

Suppose instead that the external magnetic field for the disk dynamo were still spatially uniform, but varied with sinusoidally with time, say $\mathbf{B}_{\text{ext}} = B_0 \cos \omega' t \, \hat{\mathbf{z}}$.

This example was used in Cohn in [324] as an argument that generalized flux rule eq. (3) does not always work. It was then pointed out by Bewley [332] that the time-dependent magnetic field induces eddy currents, which induce additional magnetic fields, which induce yet more electric fields, ... That is, this example cannot be well analyzed without first solving for the total electric and magnetic fields, and total currents, in the absence of the external circuit elements ABCD, which depend on the conductivity of the copper disk.

For very high conductivity, the induced magnetic field of the eddy currents cancels the external magnetic field inside (and at the flat surfaces of) the copper disk,[26] in which case no $\mathcal{EMF}$ is developed in the circuit.

2.4.2 Hering's Experiments

In 1908 and 1915, Hering [272, 285, 551], reported experiments sketched below, in which the circuits were deformed with time so as to link magnetic flux initially, but not later.[27] However, no $\mathcal{EMF}$ was observed by the galvanometer.

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[26]Recall that a superconductor expels an external magnetic field [299].

[27]A precursor of Hering's experiments was discussed by Neumann on p. 260 of [213].
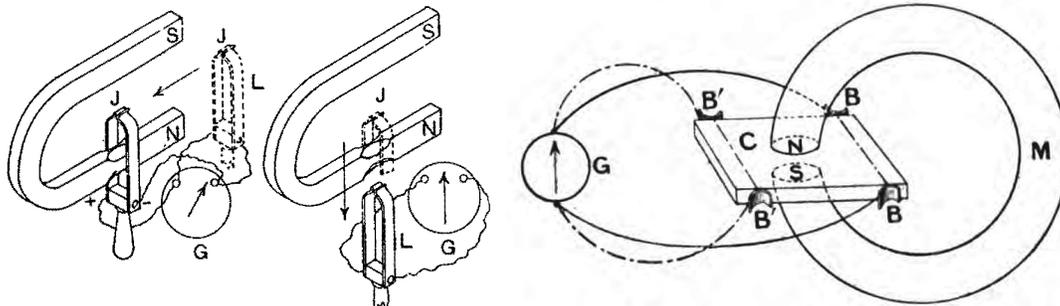
In the experiment on the left, the circuit containing spring clips was moved downwards until they no longer surrounded the N pole of the (conducting) magnet. In the experiment on the right, the copper plate was fixed with respect to the magnet, while sliding contacts moved from the right to the left of the magnet poles.[28]

Hering's first experiment is also discussed in [367, 377, 380]; the figure below is from [380]. The galvanometer remains fixed in the lab, while the leads are moved past the fixed (conducting) magnet such that a complete circuit always exists; alternatively, the magnet is moved towards the galvanometer while the tips of the leads slide around it.
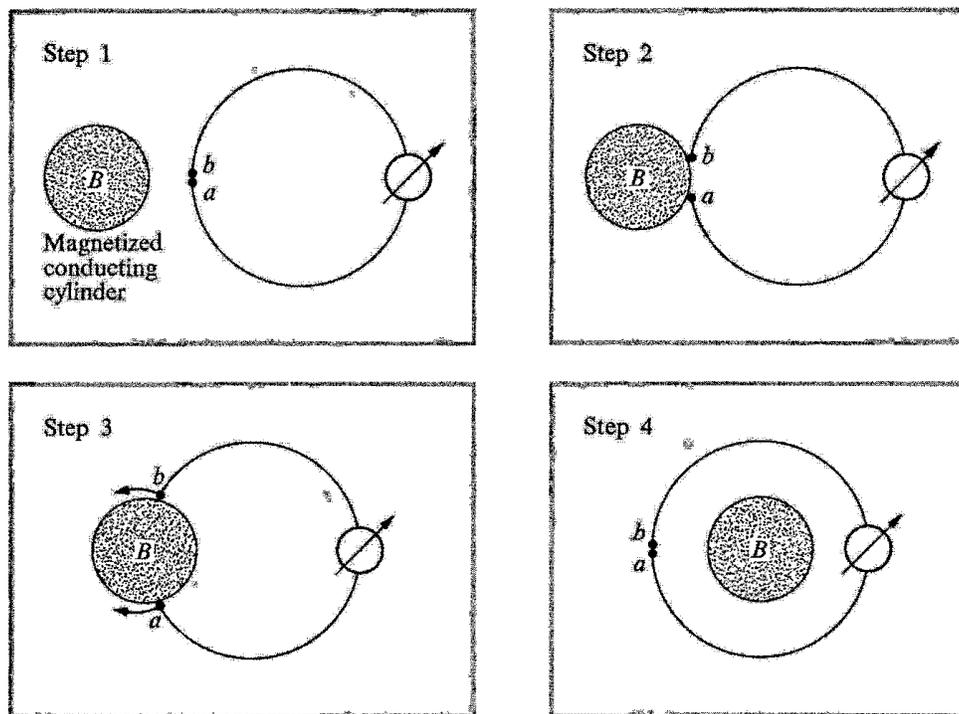

FIGURE 15.6

An experiment involving a change in flux through a circuit, without generation of an emf.

Analysis of this example is complicated by the ambiguity as to where the circuit lies. If we suppose in the experiment above (Fig. 15.6) the circuit in steps 2 and 3 lies along the

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[28]This experiment was described by Faraday (1831) in Art. 101 of [81], but with the connections to the copper plate fixed rather than sliding. In that case, the galvanometer reading was nonzero.

line joining points a and b, then the flux linked by the circuit is time dependent, and eq. (3) would predict a nonzero $\mathcal{EMF}$ during the time that points a and b are not coincident.

On the other hand, if we suppose that when point a and b do not touch the circuit is completed around the circumference of the magnet and to its left, then the linked flux jumps discontinuously at the moment when a and b first separate.

Another alternative is to suppose that when point a and b do not touch the circuit is completed around the circumference of the magnet to the right, then the linked flux is always zero (although the length of the circuit drops discontinuously by the circumference of the magnet at the moment that points a and b touch again).

If we insist on use of generalized flux law (3), then we must suppose only the third alternative is the "correct" one.

This choice is in agreement with Carter's Rule of sec. 2.3, while the first circuit is not, and the second circuit is excluded by the additional proviso of sec. 2.3.

In contrast, use of eq. (9) immediately predicts zero for the case of the magnet always at rest, noting that the velocity $\mathbf{v}$ in the computation of the motional $\mathcal{EMF}$ is the velocity of the conductor at a point on the circuit, rather than the velocity of the circuit itself. For the case that the magnet moves towards the galvanometer, both the $\mathcal{EMF}_{\text{fixed loop}}$ and the motional $\mathcal{EMF}$ are nonzero, but equal and opposite, such that the total $\mathcal{EMF}$ is again zero (see p. 883 of [367]).

If one supposes in Hering's experiment, on the right of the figure on p. 6, that the circuit lies along a straight line inside the copper between the two sliding contacts, then the generalized flux law (3) predicts a nonzero $\mathcal{EMF}$, at least when that line passes between the poles of the permanent magnet.

On the other hand, if one supposes that the circuit lies on the two "sides" of the copper plate plus its left edge, then the flux through this circuit is always zero, and no $\mathcal{EMF}$ is predicted. This choice of the circuit is consistent with Carter's Rule of sec. 2.3.

Alternatively, whatever path of the circuit is considered, both $\mathcal{EMF}_{\text{fixed loop}}$ and $\mathcal{EMF}_{\text{motional}}$ are zero, so eq. (9) predicts zero $\mathcal{EMF}$.[29],[30]

2.4.3 Tilley's Example

In 1967, Tilley [382] discussed a variant of Hering's examples: no $\mathcal{EMF}$ is generated in the circuit shown below, which changes discontinuously in a manner that changes the linked flux. The result violates the generalized flux rule (3) unless one accepts the proviso to Carter's Rule in sec. 2.3, and is also correctly explained by eq. (9).

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[29]We can also consider a variant of Hering's second experiment in which the leads and sliding contacts remain fixed in the lab, but the copper plate moves. Here, $\mathcal{EMF}_{\text{fixed loop}} = -\mathcal{EMF}_{\text{motional}}$, so eq. (9) predicts zero $\mathcal{EMF}$.

According to Carter's Rule for use of eq. (3), we must consider a circuit that passes along the edge of the moving copper plate, not the path directly between the sliding contacts which is at rest in the lab. Then, we also predict the $\mathcal{EMF}$ to be zero.

[30]As stated about such problems by Laithwaite on p. 99 of [377]: *Failure to get the answers correct without actually carrying out the experiments leaves one with the feeling that something was neglected in one's education; but to predict all answers with certainty involves, even for the expert, a considerable amount of mental gymnastics.*
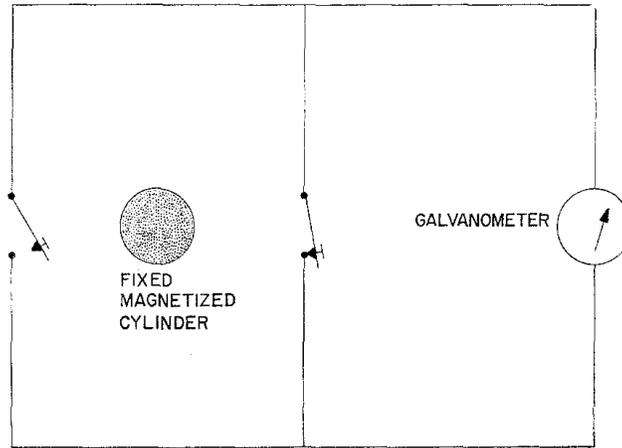
Fig. 1. When the switch on the left is closed and that on the right is opened, the galvanometer circuit experiences a large flux change but there is no induced electromotive force.

2.4.4 Blondel's Experiment

In 1914, Blondel [283, 284, 515] performed an experiment to illustrate the difficulty in applying the generalized flux rule (3) to circuits whose shape changes with time.

As sketched in the figure below, a region of static magnetic field (perhaps due to a cylindrical, permanent magnet) was surrounded by a coil of wire that unwound from a spool on the right onto a (nonconducting) cylindrical drum which was coaxial with, but outside, the magnetic field. A galvanometer made sliding contact with the wire at point C, and with a circular, conducting ring R (to which one end of the wire is attached) at point F.
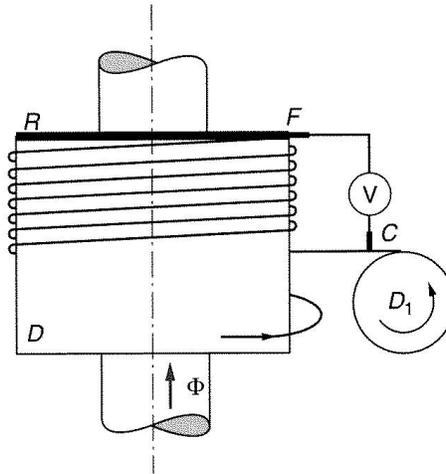
In this example, both $\mathcal{EMF}_{\text{fixed loop}}$ and $\mathcal{EMF}_{\text{motional}}$ are zero for the circuit containing the galvanometer and the turns of the wire on the drum, so the total $\mathcal{EMF}$ is zero according to eq. (9). And indeed no $\mathcal{EMF}$ was observed by the galvanometer.

However, because the number of turns of wire on the drum increased with time, the magnetic flux through the circuit increased with time, so eq. (3) predicts a nonzero $\mathcal{EMF}$.

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This appears to be a failure of the generalized flux rule (3).[31]

As pointed out by Bewley, pp. 15-18 of [370], unless the point where the wire is attached to the circular ring R is in line with the sliding contact F, there are actually two paths for the current to flow in that ring, and the notion of the flux through the circuit is not well defined. Hence, it is not appropriate to try to apply the generalized flux rule to this example.

In other words, the flux through the circuit changes discontinuously whenever the point where the wire is atttached to the circular ring R passed the sliding contact with the ring. Hence, according to Carter's Rule (sec. 2.4 above) the generalized flux rule (3) does not apply here.

Bewley introduced the concept of **substitution of circuits** (of which Hering's, Tilley's and Blondel's circuit are examples), and salvaged the generalized flux rule (3) by noting that there is no induced $\mathcal{EMF}$ in these cases. But in the present author's view, the fact that a straightforward application of eq. (9) to such examples shows that there is no $\mathcal{EMF}$ indicates a superiority of that approach over use of the generalized flux rule.

2.4.5 Cullwick's Experiment
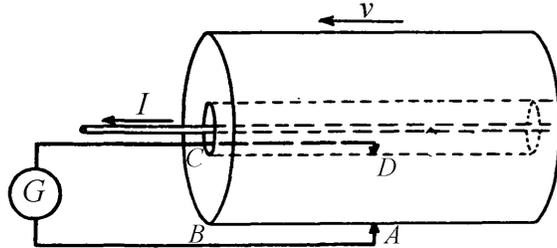
In 1939, Cullwick [310] reported an experiment, sketched on the previous page, in which a steady current I flowed in a fixed wire along the axis of an annular metal cylinder that moved with velocity v parallel to its axis (see also sec. 9.6 of [354]). A galvanometer was connected to contacts on the inner and outer radii of the cylinder, which contacts remain fixed in the lab as the cylinder slid past them.

Cullwick found that reading of the galvanometer did not depend on the permeability of the cylinder, which could be made of copper or of iron

The (azimuthal) magnetic field B_{in} (at a given distance from the axis) inside a cylinder of (relative) permeability μ is μ times the field B_{out} outside the cylinder (at the same radial distance); $\mathbf{B}_{\text{in}} = \mu \mathbf{B}_{\text{out}}$.

As in Hering's example, there is an ambiguity as to the path of the circuit, which might be on a straight line from A to D inside the annular cylinder between the sliding contacts, or along some other line that connects these two points.

Applying the generalized flux rule (3) to the circuit that includes the straight path from A to D implies a (changing) flux through the circuit that depends on the permeability of the cylinder, in contradiction to the experimental result.

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[31]Faraday's own version of his law, that the induced $\mathcal{EMF}$ depends on the "cutting" of lines by the magnetic field, readily predicts there to be no $\mathcal{EMF}$ in Blondel's experiment, since the magnetic field is zero everywhere along the wire.

However, this path is not to be considered, according to Carter's Rule of sec. 2.3, and instead we should use path $ABCD$ along the surface of the annular cylinder, in which case the (changing) flux linked by the circuit is entirely due to the magnetic field outside the cylinder, which is independent of the permeability of the cylinders.

Thus, successful use of the generalized flux rule (3) requires use of the path $ABCD$.

In contrast, if we use an analysis based on the partition (9), we get a correct result using path AD (as well as with path $ABCD$). In this case $\mathcal{EMF}_{\text{fixed loop}}$ is proportional to $(B_{\text{in}} - B_{\text{out}})v$, which is larger with an iron cylinder than with a copper one, *i.e.*, $\mathcal{EMF}_{\text{fixed loop}}$ is zero for a moving copper cylinder. Meanwhile, there exists a motional $\mathcal{EMF}$ along path AD (and also along $ABCD$), proportional to vB_{in}, such that the total $\mathcal{EMF}$ in the galvanometer circuit is proportional to $(B_{\text{in}} - B_{\text{out}})v - vB_{\text{in}} = -vB_{\text{out}}$, and is independent of the permeability μ, as observed by Cullwick in lab experiments.

Thus, while Cullwick's example can be analyzed using the generalized flux rule (3) if one somehow intuits that path $ABCD$ is to be used, the partition (9) is more robust against the ambiguity as to the path of the current.[32]

2.4.6   Circular Loop that Expands or Contracts Radially

This example was mentioned by Franklin (1908), p. 1357 of [272], as part of the discussion following an argument by Hering that the existence of a motional $\mathcal{EMF}$ does not always imply that the total $\mathcal{EMF}$ is nonzero.

We consider a circular loop of time-dependent radius $a(t)$ in a uniform magnetic field $\mathbf{B}$ perpendicular to the plane of the loop. In case that the loop is not a conductor, the electric field $\mathbf{E}$ can be zero everywhere.

Then, a naïve use of eq. (2) would lead to the inference that

$$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\mathbf{B} \cdot \frac{d\mathbf{Area}}{dt} = -2\pi a B \frac{da}{dt}, \tag{17}$$

which is nonzero, and hence the electric field must be nonzero also, in contradiction to our assumption. However, this is a misuse of the "flux law".

A better approach is to consider eq. (3) or eqs. (5)-(6). Equation (5) tells us that,

$$\mathcal{EMF}_{\text{fixed loop}} = -\int_{\text{loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = 0, \tag{18}$$

and eq. (6) gives,

$$\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = -2\pi a B \frac{da}{dt}. \tag{19}$$

while eq. (3) implies,

$$\mathcal{EMF}_{\text{total}} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\mathbf{B} \cdot \frac{d\mathbf{Area}}{dt} = -2\pi a B \frac{da}{dt} = \mathcal{EMF}_{\text{fixed loop}} + \mathcal{EMF}_{\text{motional}}. \tag{20}$$

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[32]Bewley used his concept of *substitution of circuits* to analyze Cullwick's example via the generalized flux law on pp. 71-77 of [370].

The story is consistent, that an imaginary circular loop has an imaginary $\mathcal{EMF}$ if it expands or contracts while in a uniform magnetic field and zero electric field.

There now exist conducting rubber bands, so this example could be realized with a physical, conducting loop, in which case the $\mathcal{EMF}$ (20) would become physical, and a current would flow in the contracting or expanding rubber band.

2.4.7 A Two-Turn Coil

In 1949, Slepian [322, 325] pointed out that when computing the magnetic flux through a two-turn coil, one might consider the surface through which the flux passes to be a Möbius strip, *i.e.*, a one-sided surface, in which case one would deem the total flux to be zero. This reinforces the tacit assumption in Faraday's Law (3) that the surface associated with a circuit is two-sided, as in the figure on the right below (a so-called *Riemann surface*).
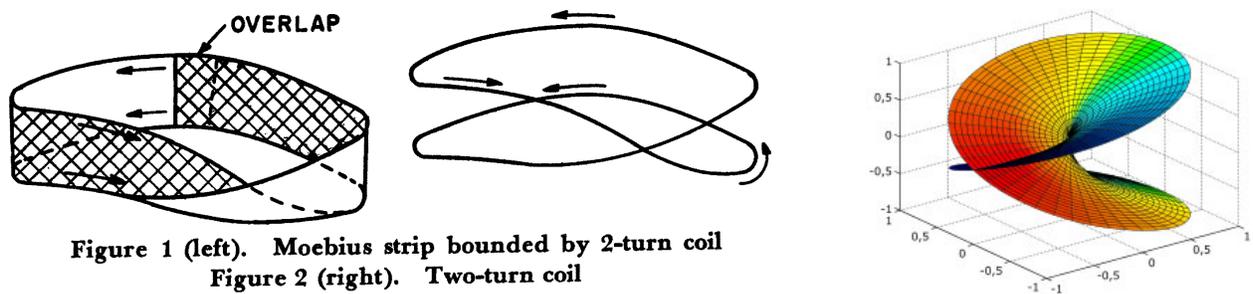

Figure 1 (left). Moebius strip bounded by 2-turn coil
Figure 2 (right). Two-turn coil

In the author's view, these examples reinforce a preference for the partition (9) over the generalized flux rule (3).

3 Additional Comments

The discovery of magnetic induction of electric effects by Faraday and others (≈ 1831) was a major addition to the unification of electricity and magnetism by Ørsted and Ampère and others (≈ 1820), and set the stage for the unification of electromagnetism and optics by Maxwell (1862). Comments on these rich phenomena can be made from many perspectives, and in this spirit we offer some additional remarks in this section, as well as a historical review in the (long) Appendix below. These comments are, of course, not entirely independent of the previous discussion in this note.

3.1 Differing Views of "Engineers" and "Physicists"

The "flux rule" issue is complicated in that the "flux rule" means different things to different people, who might loosely be characterized as "engineers" or "physicists".[33]

When considering electromagnetic effects, "physicists" tend to emphasize the "field theory" of it, as championed by Maxwell [176], in which the abstract concept of electric and magnetic fields play a major role. For "physicists", the "flux rule" is often considered to be eq. (2) which is a relation between the abstract fields **E** and **B**, independent of such

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[33]For a humorous view, see http://kirkmcd.princeton.edu/examples/EM/hickey_pw_28-12_52_15.pdf

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"engineering" details as "wires", which might or might not be in motion. In this version of the "flux rule", there is no mention of an $\mathcal{EMF}$.

In contrast, "engineers" are more interested in "practical" effects such as the behavior of electric currents in (electrical) conductors, which may well be in motion. Such currents are said to be driven by an $\mathcal{EMF}$, which can be considered to exist even when no current is flowing.

For example, "physicists" consider Ohm's law [76] for steady currents in conductors to be,[34]

$$\mathbf{J} = \sigma \mathbf{E}, \tag{21}$$

where the current density $\mathbf{J}$ is the electric current per unit area and σ is the electrical conductivity. Equation (21) describes a "local" behavior in some small region of an electrical conductor. In contrast, "engineers" consider Ohm's law (for steady currents) I to be a "global" relation for a closed loop of conducting material,

$$\mathcal{E} = I \sum R, \tag{22}$$

where the electric resistance R of a wire of conductivity σ, length l and cross sectional area A is given by $R = l/\pi A\sigma$, the magnitude of the current density is $J = I/A$, and $\mathcal{E}$ is the electromotive force ($\mathcal{EMF}$). The loop of conductor might consist of segments of different conductivities, and hence different resistances, and the "law" (22) involves the total resistance $\sum R$ of these segments.

The electromotive force $\mathcal{E}$ is difficult to define in general, which leads some "engineers" to say that an electromotive force is the same thing as the "voltage" measured by a "voltmeter" [523].

In electrostatics, electromotive force $\mathcal{E}$ can be well defined for a loop, whether conducting or not, by the ("physicist") relation,

$$\mathcal{E} = \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l}. \tag{23}$$

Furthermore, in electrostatics the $\mathcal{EMF}$ can be uniquely defined for any point b, with respect to point a, as,

$$\mathcal{E}_a(b) = \int_a^b \mathbf{E} \cdot d\mathbf{l} \ (= V_b - V_a), \tag{24}$$

independent of the path of integration from a to b 9(so long as the path source not include a source of $\mathcal{EMF}$ such as a battery; see sec. 3.5 of [523]). This $\mathcal{EMF}$ is the work done on a unit charge as is moves from a to b.[35]

The electrostatic $\mathcal{EMF}$ between two points is also called the voltage difference, or, more loosely, just the voltage.

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[34]Ohm was a "physicist", and his original statement was much closed to eq. (21) than to (22).

[35]Recall that the electric field $\mathbf{E}$ at a point $\mathbf{x}$, as introduced by W. Thomson and by Maxwell, was called the force, meaning the force that would be experienced by a unit electric charge if placed at that point (and if such placement did not perturb the sources of the field $\mathbf{E}$).

A difficulty for considerations of the "flux rule" in time-dependent situations is that there is no generally accepted definition of $\mathcal{EMF}$ here.

In general, the integral in eq. (24) depends on the path between a and b in a time-dependent situation. Of course, this integral is well defined for a particular path, so when considering an electrical circuit where a path is evident, many people suppose that eq. (23) applies (at least for that path).

In time-dependent examples, an element $d\mathbf{l}$ of a path (whether or not the path lies inside a conductor) might be associated with a velocity $\mathbf{v}$, and the force, due to the fields $\mathbf{E}$ and $\mathbf{B}$ whose sources are external to that element, on an electric charge q placed on that (possibly moving) element, is the Lorentz force,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{25}$$

Then, one might define the $\mathcal{EMF}$ associated with a particular (possibly time dependent) loop as,

$$\mathcal{E} = \oint_{\text{loop}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}, \tag{26}$$

and a path-dependent $\mathcal{EMF}$ at point b, with respect to a reference point a can be taken as,

$$\mathcal{E}_a(b) = \int_a^b (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}, \tag{27}$$

which is the work done on a unit charge as is moves from a to b.

3.2 $\mathcal{EMF}$ and Circuit Analysis

A prominent "engineering" application involving $\mathcal{EMF}$ is the analysis of time-dependent electrical circuits, most of which are entirely at rest in the lab frame. The classical analysis of Kirchhoff [115, 122]) of networks of resistors and batteries at rest with steady currents was based on the notion that a scalar voltage could be assigned to any point in a circuit. This was taken by Kirchhoff [129] to be the electric scalar potential $V = \int \rho \, d\text{Vol}/4\pi\epsilon_0$.[36]

In applications of Kirchhoff's law to time-dependent circuits, a scalar "voltage" is assumed to exist at any point in the circuit.[37] However, the concept (27) of $\mathcal{EMF}$ at point b, with respect to point a is ambiguous if the circuit contains multiple loops such that there is more than one path for current to flow from a to b, so this definition does not well translate into a scalar function at each point in the circuit (even when $\mathbf{v} = 0$). It is this author's view that the scalar voltage ($\mathcal{EMF}$) to be used in analysis of time-dependent circuits is best considered to be the electric scalar potential in the Lorenz gauge, as discussed in [523].

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[36]The electric scalar potential was introduced by Green (1828), p. 9 of [184] (who first used the term potential), following use of a scalar potential for gravity by Lagrange (1773), p. 348 of [15], by Laplace (1799), p. 25 of [19], by Poisson (1813), p. 390 of [23], and to "permanent" magnetism by Poisson (1824), p. 493 of [59], and (1826), p. 463 of [74].

[37]The use of "voltmeters" to measure the "voltage" in circuits at rest, but with time-dependent magnetic fields, can exhibit counterintuitive behavior, as reviewed in [514]. Some of this behavior is so baffling that Lewin has claimed that "Kirchhoff's Loop Rule is for the Birds",
https://www.youtube.com/watch?v=LzT_YZOxCFY.

This usage provides a tension between the notion of $\mathcal{EMF}$ as used in analysis of circuits at rest and that for circuits with moving parts (as is the theme of the present note). However, in most practical circuits at rest, eq. (24) provides a useful approximation to a unique scalar "voltage" at points at the junction between different circuit elements, so we can be optimistic that for moving circuits the definition (27) will also provide a useful approximation.

3.3 Digression on Moving Frames of Reference

We digress to consider a related issue that has led to ongoing confusion in discussion of moving circuits, namely the relation between the electromagnetic field, potentials and $\mathcal{EMF}$s in different frames of reference.

Thus far, we have only discussed these quantities in an inertial lab frame, which typically includes a "voltmeter" to measure the $\mathcal{EMF}$.

Faraday's disk dynamo (homopolar generator) involves a rotating disk, so one could consider the electrodynamics as observed in the rotating frame of the disk, although in no variant of the homopolar generator considered thus far is any measurement made in that frame. As reviewed in [507], electrodynamics in a rotating frame involves many nonintuitive features, so we consider it best to leave them out of the present discussion.[38]

However, the relation between electrodynamics in two inertial systems that have a relative velocity is well described by special relativity (which is generally considered to be implicit in Maxwell's theory). In particular, the electric field $\mathbf{E}'$ observed in an inertial frame with velocity $\mathbf{v}$ with respect to the inertial lab frame is given by,

$$\mathbf{E}' \approx \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{28}$$

when v is small compared to the speed c of light in vacuum. In this same approximation there is no Lorentz contraction of length, such that $d\mathbf{l} = d\mathbf{l}'$, so we can rewrite eq. (9) as,

$$\mathcal{EMF}_{\text{total}} = \oint_{\text{loop}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \approx \oint_{\text{loop}} \mathbf{E}' \cdot d\mathbf{l}' = \mathcal{EMF}'_{\text{flux rule}}, \tag{29}$$

where $\mathcal{EMF}'_{\text{flux rule}}$ would have to be measured by apparatus at rest in the moving frame (which is therefore different than an apparatus that measures $\mathcal{EMF}$ in the lab frame).

The relation (29) is valid, but not very relevant for any practical experiment on induction in a moving circuit. Nonetheless, papers such as [344, 384, 408, 433, 440, 481, 494] try to suggest that eq. (29) somehow changes our understanding of magnetic induction in the lab frame. These suggestions are misguided in the view of the present author[39,40]

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[38]In several papers [300, 304, 309, 312], Kron argued that if one chooses to analyze problems of rotating electrical machinery in the rotating frame, then one must employ much of the mathematical apparatus of general relativity, including use of the Christoffel symbols. For additional comments by the present author on this theme, see [528].

[39]A discussion by the author of one misguided use of relativity for electromagnetic induction is in [556].

[40]Faraday's law, plus Maxwell's version of Ampère's law, can be used to deduce the Lorentz transformations of the fields $\mathbf{E}$ and $\mathbf{B}$, as discussed in Appendix C of [535].

3.3.1 Einstein on Electromagnetic Induction

Einstein's great paper [269] that introduced his theory of special relativity began with comments on different ways of thinking about electromagnetic induction:[41]

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

He returned to this theme at the end of sec. 6 of [269] with the comments:

1. If a unit electric point charge is in motion in an electromagnetic field, there acts upon it, in addition to the electric force, an "electromotive force" which, if we neglect the terms multiplied by the second and higher powers of v/c, is equal to the vector-product of the velocity of the charge and the magnetic force, divided by the velocity of light. (Old manner of expression.)

2. If a unit electric point charge is in motion in an electromagnetic field, the force acting upon it is equal to the electric force which is present at the locality of the charge, and which we ascertain by transformation of the field to a system of co-ordinates at rest relatively to the electrical charge. (New manner of expression.)

The analogy holds with "magnetomotive forces". We see that electromotive force plays in the developed theory merely the part of an auxiliary concept, which owes its introduction to the circumstance that electric and magnetic forces do not exist independently of the state of motion of the system of co-ordinates. Furthermore it is clear that the asymmetry mentioned in the introduction as arising when we consider the currents produced by the relative motion of a magnet and a conductor, now disappears. Moreover, questions as to the "seat" of electrodynamic electromotive forces (unipolar machines) now have no point.

Einstein argued that when considering an electron, its rest frame plays a special role, and only the electric field in this frame exerts an electromagnetic force on the electron. Observers for whom the electron is in motion attribute the electromagnetic force to both the electric and magnetic fields (in their frames of reference). Hence, the notion of the "seat" of the $\mathcal{EMF}$ is observer dependent (and said somewhat harshly by Einstein to "have no point").

It remains that when considering measurements of induced currents and $\mathcal{EMF}s$ it is most relevant to use the frame of reference in which the measurement apparatus is at rest, rather than considering the rest frames of various electrons which participate in the current.

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[41]See also [495].

A   Appendix: Historical Review

A.1   Peregrinus

In 1269, Peregrinus [1] concluded from experiment that a (permanent) magnet has two "poles", and that like poles of different magnets repel, while unlike poles attract. Peregrinus' methods were later notably extended by Gilbert.

A.2   Gilbert

In 1600, Gilbert published a treatise [2], which included qualitative notions of magnetic energy and lines of force.[42] These seem to have been inspired by experiments (suggested by Peregrinus) in which magnetized needles (or steel filings, p. 162 of [2]) were used to probe the space around larger magnets, particularly spherical magnets called terrellas. The observed directions of the needle (or filings) suggested the existence of lines of force throughout space, and the ability of the magnet to deflect the needle into alignment with them suggested (to Gilbert) that some kind of magnetic energy existed outside the magnet itself.

The figures below are from pp. 122 and 247 of [2].
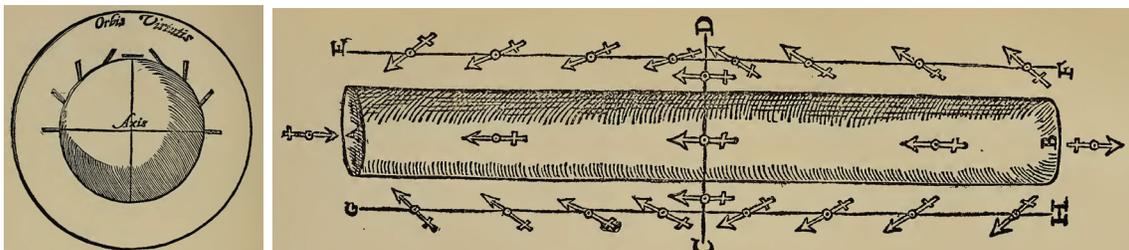
A.3   Descartes

In 1644, Descartes published a qualitative treatise on physical science [3]. Among its notable features is perhaps the earliest conception of momentum (mass times velocity, p. 59 ff),[43] and that light could be due to static pressure in a kind of elastic medium that fills all space, later called the æther (pp. 94-104, part III). And, on p. 271, part IV, he presented a figure based on use of iron filings near a magnet, which illustrates Gilbert's lines of magnetic force.


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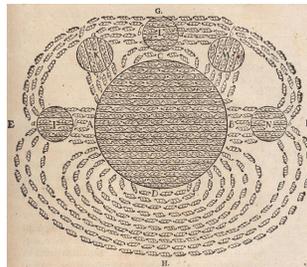
[42]Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166: *May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!*

[43]The next use of momentum may be by Wallis and Wren in 1668 [4], and by Huygens in 1669 [5].

A.4   Gray

The phenomenon of conduction of electricity may have been first reported by Gray in 1731 [6], who described experiments on the "conveyance" and "communication" of electricity (including evidence that people conduct electricity).[44]
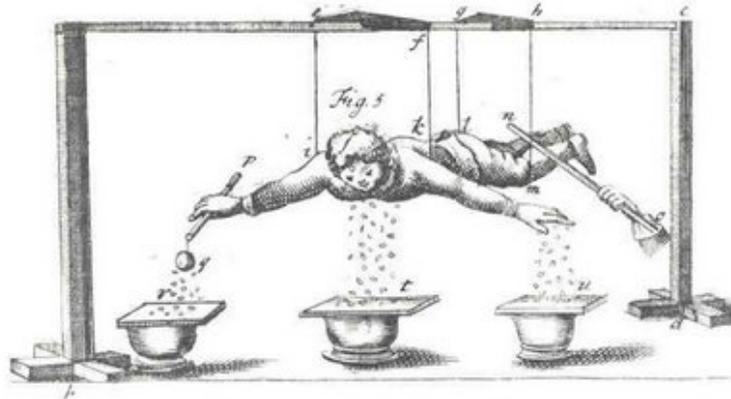
In 1739, Desaguliers [9] (a coworker of Gray) coined the term conductor, as well as the term insulator.

A.5   Michell

In 1750, Michell published a treatise on the manufacture of magnets [11], including on p. 19 the statement that the repulsive force between like poles of two magnets fall off as $1/r^2$.[45]

A.6   Aepinus

In 1759, Aepinus [12] published a long essay on electricity and magnetism that included a sketch of lines of iron filings on a plane above a horseshoe magnet. He speculated that these line formed circles, and that the magnetic force due to a single pole varied as $1/r^2$.
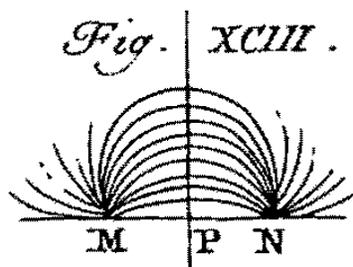
A.7   Priestley

In 1766, Priestley [14] deduced that the static force between electric charges varies as $1/r^2$, similar to the force of gravity except that like charges repel rather than attract.

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[44]For a historical survey of research into electricity in this era, see, for example, [410].

[45]Michell is also credited with being the first to discuss what are now called black holes [13].

A.8 Coulomb

In 1785, Coulomb confirmed (and made widely known) that the static force between pairs of electric charges q_1 and q_2 varies as $q_1 q_2 / r^2$ [16], and that the force between idealized magnetic poles p_1 and p_2 at the ends of long, thin magnets varies as $p_1 p_2 / r^2$ [17]. The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form.

Coulomb also noted that magnetic poles appear not to be isolatable, conjecturing (p. 306 of [211]) that the fundamental constituent of magnetism, a *molécule de fluide magnétique*, is a dipole, such that effective poles appear at the ends of a long, thin magnet.
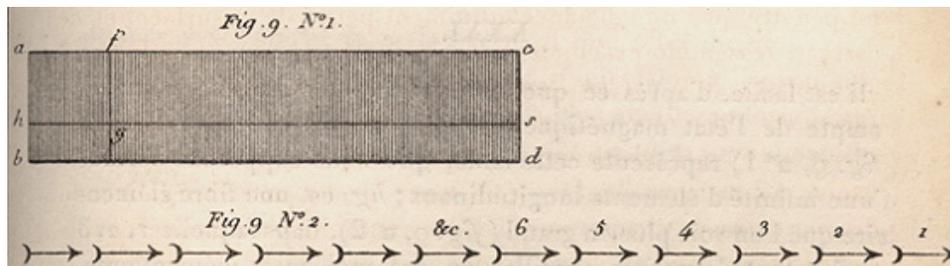
Coulomb's argument is much superior to that by Gilbert, p. 247 of [2].

A.9 Poisson

In 1812, Poisson [22] extended the use (by Lagrange and Laplace) of a potential $V = -Gm/r$ (= energy per unit mass) for the gravitational force between of a mass m and a unit test mass to the case of static electrical forces, and in 1824-26 for static magnetic forces [59, 74].[46]

A.10 Ørsted

In 1820, Ørsted [24]-[27],[41] published decisive evidence that electric currents exert forces on permanent magnets and *vice versa*, indicating that electricity and magnetism are related.[47] Ørsted's term "electric conflict", used in his remarks on p. 276 of [25], is a precursor of the later concept of the magnetic field:

It is sufficiently evident from the preceding facts that the electric conflict is not confined to the conductor, but dispersed pretty widely in the circumjacent space. From the preceding facts we may likewise infer that this conflict performs circles.

Ørsted also argued from Newton's 3rd law of action and reaction that a magnet should exert a force on an electric current, and published evidence for this in [28].

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[46]Following the precedent from gravity, Poisson did not appear to ask where the configuration energy, such as $q_1 q_2 / r$ and $p_1 p_2 / r$, resided, nor did he consider the quantity $-\boldsymbol{\nabla} V$ to be a force field in the space outside the relevant charges or poles.

[47]Reports have existed since at least the 1600's that lightning can affect ship's compasses (see, for example, p. 179 of [182]), and an account of magnetization of iron knives by lightning was published in 1735 [8]. In 1797, von Humboldt conjectured that certain patterns of terrestrial magnetism were due to lighting strikes (see p. 13 of [498], a historical review of magnetism). A somewhat indecisive experiment involving a voltaic pile and a compass was performed by Romagnosi in 1802 [470].

Historical commentaries on Ørsted's work include [346, 351, 541].

A.11 Biot and Savart

Among the many rapid responses in 1820 to Ørsted's discovery was an experiment by Biot and Savart [30, 33] on the force due to an electric current I in a wire on one pole, p, of a long, thin magnet. The interpretation given of the result was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [38, 62] with a form that can be written in vector notation (and in SI units) as,

$$\mathbf{F}_{\text{on } p} = \frac{\mu_0 \, p}{4\pi} \oint \frac{I \, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2},\tag{30}$$

where $\mathbf{r}$ is the distance from a current element $I \, d\mathbf{l}$ to the magnetic pole. There was no immediate interpretation of eq. (30) in terms of a magnetic field, $\mathbf{B} = \mathbf{F}/p$, due to the electric current I,[48]

$$\mathbf{B} = \frac{\mu_0}{4\pi} \oint \frac{I \, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2},\tag{31}$$

which expression is now commonly called the Biot-Savart law.[49]

Biot and Savart did not discuss the force on an electric current, but if we suppose the force on pole p due to the electric current I is equal and opposite to the force of the pole on the electric current, and note that the magnetic field of the pole at the current element $I \, d\mathbf{l}$ is $\mathbf{B} = -\mu_0 p \, \hat{\mathbf{r}}/4\pi r^2$ for $\hat{\mathbf{r}}$, then,

$$\mathbf{F}_{\text{on } I} = \oint I \, d\mathbf{l} \times \mathbf{B},\tag{32}$$

which is now often called the Biot-Savart force law.[50] Finally, we can consider the magnetic field $\mathbf{B}$ in eq. (32) to be that due to a second loop with steady current, as in eq. (31) with $I \to I_2$, such that the combination of these two equation gives the force of one circuit on another, when they both carry steady currents. See also eq. (39) below.

A.12 Ampère

Between 1820 and 1825 Ampère made extensive studies [31, 32, 34, 36, 37, 39, 40, 47, 48, 49, 52, 53, 60, 61, 67, 69, 70] of the magnetic interactions of electrical currents.[51] Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents.[52,53]

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[48]Although the concept of the magnetic field is latent in discussions of magnetic force by Michell, Coulomb, Poisson and Ørsted (and many other in the years 1820-45), the first use of the term "magnetic field" seems to be due to Faraday, Art. 2247 of [119] (1845).

[49]An early example is on p. 220 of [268].

[50]The earliest description of eq. (32) as the Biot-Savart force law may be in sec. 2 of [270], and in English, sec. 7-6 of [360]. In France, eq. (32 is often called the Laplace law, although Laplace's (unpublished) contribution seems only to be that the magnetic force on a pole varies as $1/r^2$, as in eq. (30). See p. 339 of [293], p. 23 of [374], p. 15 of [472].

[51]Discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in [374, 452, 539]. Historical surveys of 19th-century electrodynamics are given in [276, 472], and studies with emphasis on Ampère include [359, 373, 414, 418, 420, 424, 427, 428, 541]. See also sec. IIA of [479].

[52]See p. 166 of [34], and also [420].

[53]The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampèrian (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [405, 538].

In 1822-1823 (pp. 21-24 of [70]), Ampère examined the force between two circuits, carrying currents I_1 and I_2, and inferred that this could be written (here in vector notation) as,

$$\mathbf{F}_{\text{on }1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on }1}, \quad d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) - 2\, d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on }2}, \quad (33)$$

where $\mathbf{r} = \mathbf{l}_1 - \mathbf{l}_2$ is the distance from a current element $I_2\, d\mathbf{l}_2$ at $\mathbf{r}_2 = \mathbf{l}_2$ to element $I_1\, d\mathbf{l}_1$ at $\mathbf{r}_1 = \mathbf{l}_1$.[54,55] The integrand $d^2\mathbf{F}_{\text{on }1}$ of eq. (33) has the appeal that it changes sign if elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton's third law.[56] However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton's gravitational force, and Coulomb's law for the static force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, *i.e.*, a pair of moving electric charges.[57]

An important qualitative consequence of eq. (33) is that parallel currents attract, and opposite current repel.

Around 1825, Ampère noted, p. 214 of [69], p. 29 of [70], p. 366 of the English translation in [539], that for a closed circuit, eq. (33) can be rewritten as,[58]

$$\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 I_1 d\mathbf{l}_1 \times \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \quad (39)$$

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[54]Ampère sometimes used the notation that the angles between $d\mathbf{l}_i$ and $\mathbf{r}$ are θ_i, and the angle between the plane of $d\mathbf{l}_1$ and $\mathbf{r}$ and that of $d\mathbf{l}_2$ and $\mathbf{r}$ is ω. Then, $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = dl_1\, dl_2(\sin\theta_1 \sin\theta_2 \cos\omega + \cos\theta_1 \cos\theta_2)$, and the force element of eq. (33) can be written as,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2\, dl_1\, dl_2(\cos\theta_1 \cos\theta_2 - 2\sin\theta_1 \sin\theta_2 \cos\omega)\frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on }2}. \quad (34)$$

[55]Ampère also noted the equivalents to,

$$d\mathbf{l}_1 = \frac{\partial \mathbf{r}}{\partial l_1}\, dl_1, \qquad \mathbf{r} \cdot d\mathbf{l}_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1}\, dl_1 = \frac{1}{2}\frac{\partial r^2}{\partial l_1}\, dl_1 = r\frac{\partial r}{\partial l_1}\, dl_1, \qquad d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2}\, dl_2, \qquad \mathbf{r} \cdot d\mathbf{l}_2 = -r\frac{\partial r}{\partial l_2}\, dl_2, (35)$$

where l_1 and l_2 measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = -d\mathbf{l}_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2}\, dl_2 = -\frac{\partial}{\partial l_2}(\mathbf{r} \cdot d\mathbf{l}_1)\, dl_2 = -\frac{\partial}{\partial l_2}\left(r\frac{\partial r}{\partial l_1}\right) dl_1\, dl_2 = -\left(\frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2} + r\frac{\partial^2 r}{\partial l_1 \partial l_2}\right) dl_1\, dl_2, \quad (36)$$

and eq. (33) can also be written in forms closer to those used by Ampère,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2\, dl_1\, dl_2 \left[2r\frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2}\right]\frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2 I_1 I_2\, dl_1\, dl_2 \frac{\partial^2 \sqrt{r}}{\partial l_1 \partial l_2}\frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\text{on }2}. \quad (37)$$

[56]In view of this, Maxwell called Ampère the "Newton of electricity" in Art. 528 of [191].

[57]If we follow Ampère in defining a "current element" as being electrically neutral, which is a good (but not exact [517]) approximation for currents in electrical circuits, then a moving charge is not a "current element", and such elements cannot exist except in closed circuits (contrary to remarks such as in [434]).

[58]Note that for a fixed point 2, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r} \cdot \hat{\mathbf{r}} = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr)\, dr = (df/dr)\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (39) is a perfect differential with respect to $\mathbf{l}_1$. Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write,

$$\mathbf{F}_{\text{on }1} = -2\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2}\, \hat{\mathbf{r}} = -\mathbf{F}_{\text{on }2} = -\boldsymbol{\nabla}U \qquad \text{with} \qquad U = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}, \quad (38)$$

where the magnetic interaction energy U wa first given by Neumann, p. 8 of [116].

in vector notation (which, of course, he did not use). Ampère made very little comment on this result,[59] nor factorized it into the forms now associated with the Biot-Savart law(s) (31)-(32).

In 1826, Ampère gave lectures that included discussion of the force on a magnetic pole due to an electric current, noting that the line integral of the tangential force around a closed loop is proporptional to the electric current that passes through the loop, independent of the shape of the loop [407, 509]. This was a statment of what is now called "Ampère's (circuital) law".[60]

Ampère performed an experiment in 1821-22 [48, 54, 83] that showed a weak effect of electromagnetic induction, which was largely disregarded at the time.[61]

A.13   The First Electric Motors

That electromagnetic interactions could led to rotary motion (= electric motor) was demonstrated by Faraday in 1821 [44, 51], as in the left two figures below. This was shortly followed by the device of Barlow [50] (1822; see also [56]), right figure below, which has much the form of Faraday's later disk dynamo (sec.A.17.2 below). The rapid proliferation of electromechanical devices thereafter is illustrated, for example, in [109] (1842).[62]
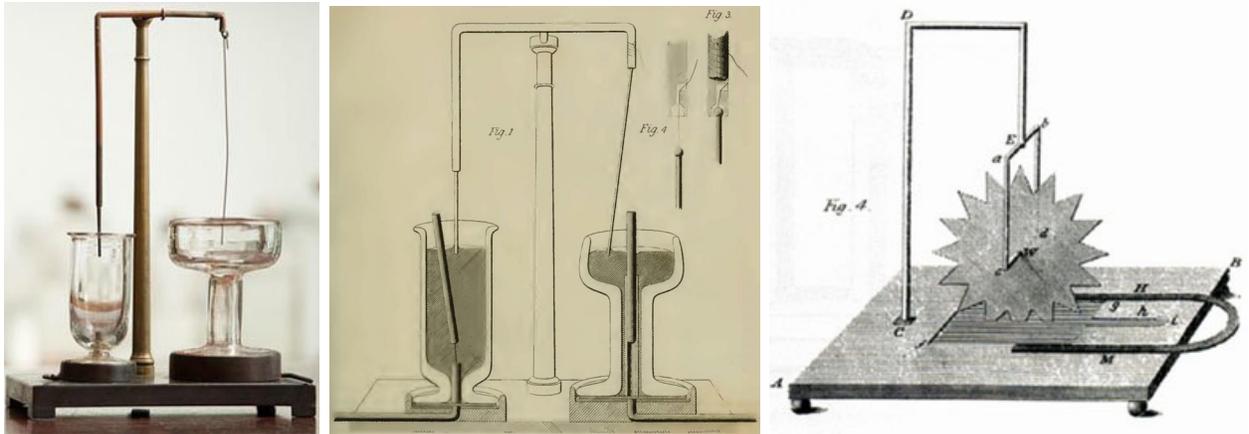
A.14   Arago

A first step towards the inverse effect, that motion of a conductor in a magnetic field produces electrical effects, was made by Arago [63, 64, 73], who reported in 1824: *the results of some experiments that he has conducted on the influence that metals and many other substances exert on a magnetic needle, which has the effect of rapidly reducing the amplitude of the*

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[59]As a consequence, the form (39) is generally attributed to Grassmann [112], as in [479], for example. See also sec. A.21 below.

[60]This "law" was noted by Maxwell on p. 56 of [158], who deduced from it that $\nabla \times \mathbf{H} = \mathbf{J}$. See also [157], and Art. 498 of [248]

[61]Reviews of this experiment include [258, 373, 423, 424].

[62]However, Faraday did little further work on magnetism until after the death of his boss, Davy, in 1829.

oscillations without altering significantly their duration.[63]

We now understand that Arago observed the effect of eddy currents due to the electric field induced by a time-dependent magnetic field inside a conductor, where the energy dissipated by Joule heating damped the kinetic energy/motion of the system. A step towards this understanding was made by Christie (1826) [72], who noted that if the copper disk is cut into two or more concentric rings, the effect of a magnet on its rotation is greatly reduced.

A.15 Ohm

In 1827, Ohm published a treatise [76] containing his famous law,[64] in a form closer to,

$$\mathbf{J} = \sigma \mathbf{E}, \tag{40}$$

where $\mathbf{J}$ is the electric current density and $\mathbf{E}$ is the electric field, both inside the rest frame of a medium with electrical conductivity σ, than to more familiar form, $\mathcal{E} = IR$, where $\mathcal{E}$ is the $\mathcal{EMF}$/potential difference across an electrical resistance R that carries electric current I.[65]

Ohm did not define a conductor so much as provide a model for it, with a flavor that electric current is related to the motion of particles. This view became characteristic of the German school in the mid 1800's, but was not taken up by the English or French until much later.

A.16 Fechner

Fechner (1831), p. 225 of [80], may have been the first to interpret the symbol $\mathcal{E}$ in Ohm's law [76],

$$\mathcal{E} = IR, \tag{41}$$

as the *electromotorische Kraft*, *i.e.*, electromotive force. Ohm and Fechner studied only steady currents in fixed loops, in which case the electromotive "force" between two points a and b is equal to the work done by electric effects when moving a unit electric charge from between two points a and b,[66]

$$\mathcal{E}(a, b) = \int_a^b \mathbf{E} \cdot d\mathbf{l} \ \left(= V(a) - V(b) \right), \tag{42}$$

---

[63]As reported by Babbage and Herschel (1825) [66]: *The curious experiments of M. Arago described by M. Gay Lussac during his visit to London in the spring of the present year (1825), in which plates of copper and other substances set in rapid rotation beneath a magnetized needle, caused it to deviate from its direction, and finally dragged it round with them, naturally excited much attention.*

They extended Arago's study by showing that if a magnet is moved with respect to a copper disk initially at rest, the disk can be set in motion.

[64]For a history of studies leading to Ohm's law, see [296].

[65]Ohm wrote of an *elektroskopsiche Kraft* (electroscopic force) in a manner that caused much confusion until Kirchhoff (1849) [129] clarified that this is the same as the electrostatic potential. Ohm's description of the difference in the electroscopic force at two points as a *Spannung* (tension) survives in the lore of circuits to the present day. See also [431].

[66]This insight appears to be due to Kirchhoff (1849) [129], following Helmholtz' great paper of 1847 [124] on the importance of energy concepts in physics.

and the sum of the $\mathcal{EMF}$s around a closed loop is zero.

Fechner was an advocate of the two-fluid theory of electric current,[67] and in 1845 [114], he argued that electric current consists of flows of equal and opposite electric charges with equal and opposite velocities.

A.17 Faraday

In this review of Faraday's studies of electromagnetic induction, we seek to understand to what extent Faraday anticipated a version of the "flux rule".

A.17.1 Evolution of Electricity from Magnetism

Faraday's first report, Arts. 27-28 of [81] (1831),[68] of an effect of magnetic induction was via an iron ring with two coils wound upon it, as in Fig. 1 below. On connecting or disconnecting one coil to/from a battery, a transient current was observed in the other coil.[69]

Photo of Faraday's original trans-
former (courtesy MIT Burndy Library).

This was a "transformer" effect, and involved no motion of conductors relative to magnetic fields.

Faraday made no further studies of transformers (which were pursued more by Henry, sec. A.18 below), and mainly studied the induction of current by conductors moving in a magnetic field over the next 20 years.[70] He gave no explanation of transformer action as an effect of time-dependent magnetic flux (or number of magnetic field lines).

In 1834 [94, 95, 96], Faraday returned to a theme mentioned briefly in Art. 32 of [81], that a spark occurs when contact is made or broken between a battery and a wire, particularly if the wire forms a coil. In Art. 1077 of [96] he noted that the spark cannot be simply related to the mechanical momentum of the electric current, since the strength of the spark depends on the shape of the loop, for a fixed length of wire.

Art. 1108 may be as close as Faraday came to relating electromagnetic induction in fixed circuits to a variation in the magnetic flux (number of magnetic field lines) through a circuit: *From the facility of transference to neighbouring wires, and from the effects generally, the inductive forces appear to be lateral, i.e. exerted in a direction perpendicular to the direction*

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[67]The two-fluids theory of electricity was first stated by du Fay (1733) [7]. The one-fluid theory was first stated by Franklin (1847) [10].

[68]Faraday's paper [81] was one of the first to have been reviewed by referees in the modern sense [442]. The historical context of this paper is reviewed in [373, 533].

[69]It is difficult to keep tracks of the signs of the currents in such examples, and Faraday struggled with these in the draft of his paper [81], as recounted in [445].

[70]A paper [100] from 1836 features a version of Faraday's iron-ring, used as a voltage step-up transformer. While the may be only the second published paper to discuss transformer action, its wording suggests that this effect was already well known.

of the originating and produced currents: and they also appear to be accurately represented by the magnetic curves, and closely related to, if not identical with, magnetic forces.

However, Art. 1114 indicates that Faraday's views on this were not very clear.

A.17.2 Extensions of the Arago Effect

In 1831, Faraday discovered that the effect of Arago could drive an electric current in an external circuit with sliding contacts to a copper disk that rotated between the poles of a permanent magnet, as sketched below (from Art. 99, p. 381, Oct. 28, 1831, of [298]). His results from studies of variants of Arago's experiment were reported in Arts. 81-139 of [81]. Art. 88 described the "disk dynamo" sketched on the following page (the version on the right is from [530]).
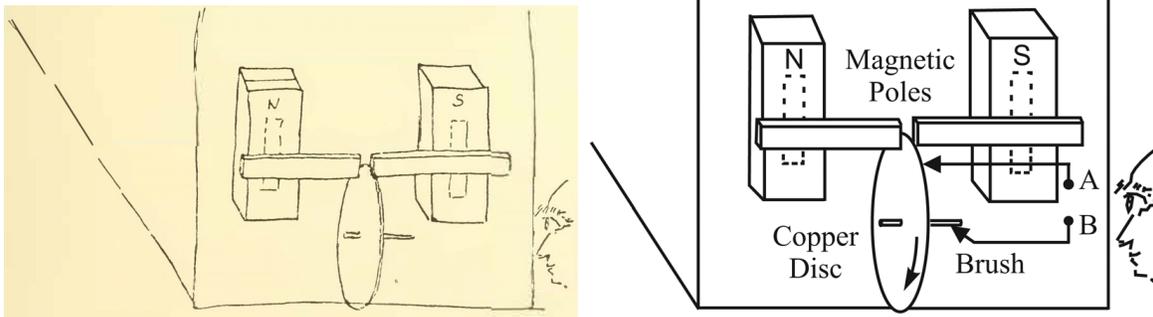
We now embark on discussion of physical explanations of these and other related studies to be considered below, noting a comment on p. 75 of [291] that:

In the history of the development of the subject there has been a singular freedom from differences of opinion as to the experimental results, but at the same time a singular lack of agreement as to the way these results were to be interpreted.

A.17.3 Eddy Currents

In Art. 131 of [81], Faraday stated:

Future investigations will no doubt ... decide the point whether the retarding or dragging action spoken of (by Arago) *is always simultaneous with electric currents,*

i.e., eddy currents. This statement might have been clearer if Art. 131 had been followed by Art. 123, where it was stated:

These currents are discharged or return in the parts of the plate on each side of and more distant from the place of the pole, where, of course, the magnetic induction is weaker.
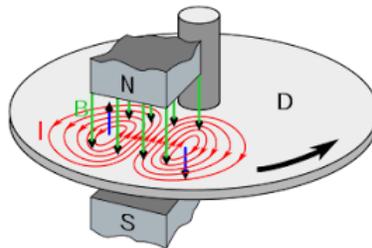
One can reasonably infer that Faraday had a vision of eddy currents as shown above, although Faraday himself never published such a sketch.[71]

A.17.4   Anticipation of the Biot-Savart/Lorentz Force Law

In Art. 99 of [81], Faraday gave a first interpretation of the behavior he had observed using a galvanometer connected to a rotating copper disk via sliding contacts:
The relation of the current of electricity produced, to the magnetic pole, to the direction of rotation of the plate, &c. &c., may be expressed by saying, that when the unmarked (south) *pole is beneath the edge of the plate, and the latter revolves horizontally, screw-fashion, the electricity which can be collected at the edge of the plate nearest to the pole is positive. As the pole of the earth may mentally be considered the unmarked pole, this relation of the rotation, the pole, and the electricity evolved, is not difficult to remember. Or if, in fig. 15* (below), *the circle represent the copper disk revolving in the direction of the arrows, and a the outline of the unmarked pole placed beneath the plate, then the electricity collected at b and the neighbouring parts is positive, whilst that collected at the centre c and other parts is negative. The currents in the plate are therefore from the centre by the magnetic poles towards the circumference.*
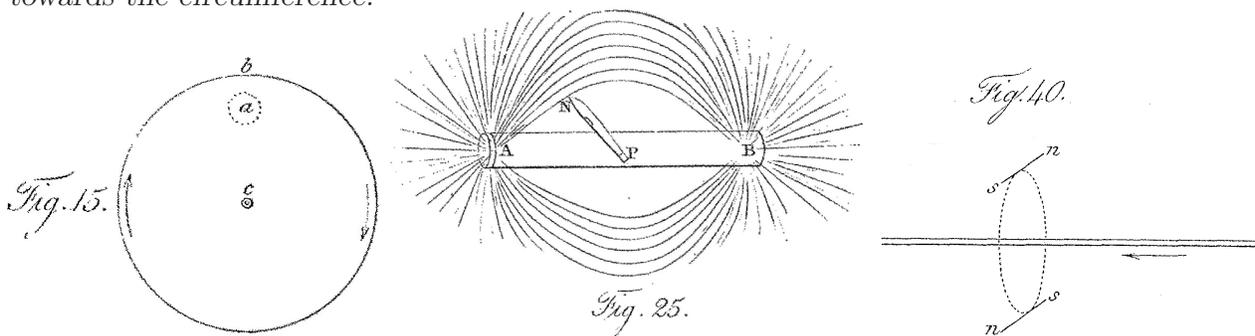
We recognize this as a version of the Biot-Savart law the force $d\mathbf{F}$ on an electric current element $I\,d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is given by vector relation,[72]

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}. \tag{43}$$

A.17.5   *Magnetic Curves* **aka Field Lines**

Faraday continued his discussion of the generation of electric currents in Arts. 114-116, referring to the *magnetic curves* in Fig. 25 above:
By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.

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[71]The notion of eddy currents is sometimes attributed to Foucault (1855) [155], and are sometimes called *Foucault currents*, although Foucault attributed their explanation to Faraday. The term "eddy current" appears in [230].

[72]Biot and Savart [30, 33, 38, 62] actually discussed the force of an electric current element on a magnetic pole p in the form $d\mathbf{F} = pI\,d\mathbf{l} \times \hat{\mathbf{r}}/r^2$, where $\hat{\mathbf{r}}$ is the unit vector from the current element to the pole. The first statement of eq. (43), the force of a magnetic pole on an electric current, may be by Maxwell (1861) in eqs. (12)-(14), p. 172 of [167] (see also sec. A.2.1 of [535]. This result was stated more crisply in Arts. 602-603 of Maxwell's *Treatise* [248]. The earliest description of eq. (43) as the Biot-Savart law may be in sec. 2 of [270].

This is Faraday's first representation of magnetic field lines.[73] In Fig. 25, A is the north pole of the magnet, and B is the south. When, the tip of a knife blade is rotated up/out of the page, with its base remaining on the magnet, Faraday noted that an electric current flows from the tip to the base, *i.e.*, from N to P, in agreement with eq. (43) and the verbal statement thereof in Art. 99.

A.17.6 Effect of "Cutting" the Magnetic Curves

In Art. 114 Faraday spoke of such action as involving a conductor *cutting* the *magnetic curves*, which notion has come to be regarded as a central feature of Faraday's vision of magnetic induction of electric currents. Now, it seems better to de-emphasize the (appealing) notion of "cutting of field lines", and rather to emphasize the interpretation of a motional $\mathcal{EMF}$ due to the Lorentz force on moving conduction electrons in a magnetic field.

That Faraday's view was close to that of motional $\mathcal{EMF}$ is illustrated in Art. 3192 of [138]. In the case of a rectangular wire loop, rotated about a median line that is perpendicular to a uniform, constant magnetic field, as sketched in Fig. 3 below, Faraday remarked:
In the first 180° of revolution round the axis a-b, the contrary direction in which the two parts c-d and e-f intersect the lines of magnetic force within the area c-e-d-f, will cause them to conspire in producing one current, tending to run round the rectangle. The parts c-e and d-f of the rectangle may be looked upon simply as conductors; for as they do not in their motion intersect any of the lines of force, so they do not tend to produce any current.

Fig. 3.
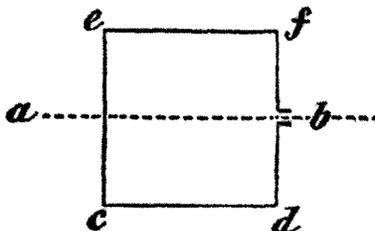
A delicacy is in the interpretation of the term *intersect*, as wire segments *c-e* and *d-f* do touch lines of force, and the number of lines they touch varies with time. One might well say that these moving wires do "cut" lines of force. However, the effect of the $\mathbf{v} \times \mathbf{B}$ is transverse to the wires, and does not drive any current. This was noted by Faraday, who must have had a good intuition as to the vector-cross-product character of the cause of the induced current.

Thus, Art. 3192 provided a clarification to earlier statements by Faraday, such as that in Art. 256 of [82]: *If a terminated wire move so as to cut a magnetic curve, a power is called into action which tends to urge an electric current through it,* which downplays the cross-product character of the "urge".

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[73]Faraday showed that the *magnetic curves* associated with a current-carrying wire are circles, Arts. 232-233 of [82] and Fig. 40 on the previous page. In this, he was recalling his discussion of *magnetic curves* in [45] (1822), and Davy's use of iron filings in studies of magnetic effects [35] (1820), following in the footsteps of Gilbert (1600), Appendix A.2, and Descartes (1644), Appendix A.3.

A.17.7 Studies Involving Cylindrical Symmetry

In 1832, Arts. 217-229 of [82], Faraday considered apparatus that was axially symmetric, and he returned to this theme in 1851, Arts. 3084-3122 of [137]. An advantage of this configuration, shown in Fig. 8 below, is that no eddy currents are induced, as belatedly remarked by Faraday (1854) in Art. 3339 of [153].

Our present view, derived in part from these studies by Faraday, is that the magnetic field of an axially symmetric magnet (with magnetization parallel to the symmetry axis) is the same in the lab frame for any value of the angular velocity of the magnet about its axis, *i.e.*, the magnetic field does not rotate along with a rotating, axially symmetric magnet.[74,75,76]

Fig. 8.   Fig. 5.   Fig. 6.
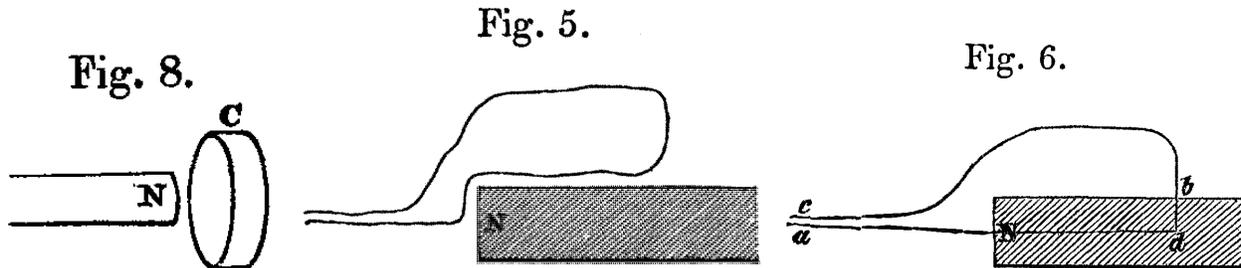
In Arts. 3093-3094, Faraday showed that when a loop of wire whose plane includes the axis of the magnet, as shown in Fig. 5 above, is rotated about the magnet, or if the loop is fixed in the lab and the magnet is rotated about its axis, then no current is induced in the loop. The latter observation is consistent with Faraday's interpretation that an induced current is generated when a conductor "cuts" lines of the magnetic field – if those lines do not rotate when the (axially symmetric) magnet rotates. Then, the former observation follows if one supposes, along with Faraday, that physics of a the rotating loop and fixed magnet is the same as that of a fixed loop and rotating magnet.[77]

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[74]In Art. 220 of [82], Faraday summarized the studies of Arts. 218-220 (repeated and extended in Arts. 3084-3122 of [137]) as:

Thus a singular independence of the magnetism and the (rotating) *bar in which it resides is rendered evident.*

[75]For a review of this theme, see [416]. Claims are still occasionally made, as in [518], that the field lines of a cylindrical magnet do rotate when the magnet rotates. Entry 11345 of Faraday's diary [303], *Do the lines of force revolve with the magnet or do they not?*, is sometimes cited, as in [546], as evidence that Faraday changed his mind on this issue, rather than merely posing a question to which his answer was no.

[76]In the first part of [82], Faraday speculated that the Earth is a rotating, permanent magnet, and that its magnetic field lines might not rotate with the Earth. However, the Earth's magnetic field is not cylindrically symmetric (see sec. 5 of [350]), and does rotate with the Earth, such that Faraday's experiments on induction by the Earth's field produced no positive results.

[77]Electrodynamics in a rotating frame is actually not quite the same as that in an inertial lab frame, as reviewed in [507] and references therein. Hence, it is generally preferable to emphasize arguments in the inertial lab frame.

In case of a planar loop of wire with uniform linear density of conduction electrons, whose plane includes the axis of an axially symmetric magnetic field $\mathbf{B}$, and which plane rotates with angular velocity $\boldsymbol{\omega}$ about that axis, the effective/motional $\mathcal{EMF}$ in the loop associated with the Biot-Savart force (43) can be computed in the lab frame, noting that the axial symmetry of $\mathbf{B}$ implies that it can be deduced from a vector potential $\mathbf{A}$ which is purely azimuthal, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A} = A_\phi(r, z)\,\hat{\boldsymbol{\phi}}$ in a cylindrical coordinate system (r, ϕ, z). Then, noting that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \omega r\,\hat{\boldsymbol{\phi}}$, such that $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} = 0$, and that $d\mathbf{Area} = d\mathrm{Area}\,\hat{\boldsymbol{\phi}}$,

$$\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = \oint_{\text{loop}} \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{l} = \oint_{\text{loop}} [\boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}] \cdot d\mathbf{l}$$

In Arts. 3095-3096, Faraday considered a planar loop of wire, part of which was inside a slot in the otherwise cylindrical magnet, as sketched in Fig. 6 above. No current was observed when the loop and the magnet were rotated together. In Art. 3091, Faraday reported on a variant of this configuration in which a current was generated when the part of the loop not inside the magnet was rotated (about the line joining the points where the loop entered and exited the magnet) while the magnet (and the part of the loop inside it) remained at rest.

Art. 3097 considered external wire segments that made sliding contact with a cylindrical magnet, as in Fig. 8 on the next page (and in Figs. 34-36, from Arts. 217-227 of [82], where Faraday had earlier studied this configuration), often called a homopolar or unipolar generator.[78] A current was generated when the external wires were rotated about the axis of the magnet when the latter remained at rest, and also when the external wires were at rest but the magnet rotated. No current was observed when the external wires and the magnet were rotated together, consistent with the null results reported is Arts. 3092 and 3095-3096.

Fig. 8.
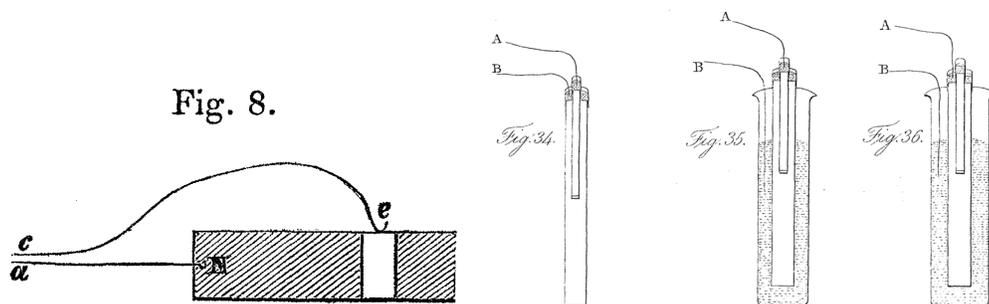
For the case that the external part of the loop was held fixed in the lab while the magnet rotated, it is important to note that part of the circuit is inside the magnet, which part is thereby rotating. In the view that the magnetic field lines do not rotate with the magnet, one might say that the part of the circuit inside the rotating magnet is also rotating, and so does "cut" field lines, inducing a current. This leads to the view expressed in [552] and elsewhere that a homopolar generator with fixed external circuit involves a loop whose shape inside the rotating conductor is changing with time, although the path of the current in the lab frame is actually time independent. It is better to say that the conduction electrons in the rotating part of the circuit experience a $\mathbf{v} \times \mathbf{B}$ Lorentz force, which drives the observed current in the loop.[79]

$$
\begin{aligned}
&= \int_{\text{surface of loop}} \boldsymbol{\nabla} \times \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{Area} = \omega \int_{\text{surface}} \boldsymbol{\nabla} \times \boldsymbol{\nabla}(r A_\phi) \cdot d\mathbf{Area} \\
&= \omega \int_{\text{surface}} \boldsymbol{\nabla} \times \left(A_\phi\,\hat{\mathbf{r}} + r\frac{\partial A_\phi}{\partial r}\,\hat{\mathbf{r}} + r\frac{\partial A_\phi}{\partial z}\,\hat{\mathbf{z}} \right) \cdot d\text{Area}\,\hat{\boldsymbol{\phi}} \\
&= \omega \int_{\text{surface}} \left(\frac{\partial A_\phi}{\partial z} + r\frac{\partial^2 A_\phi}{\partial z\partial r} - \frac{\partial A_\phi}{\partial z} - r\frac{\partial^2 A_\phi}{\partial r\partial z} \right)\,d\text{Area} = 0.
\end{aligned}
\tag{44}
$$

However, this "Maxwellian" argument is much more intricate than Faraday's.

[78]The term unipolar is due to Weber (1839) [107].

[79]This view avoids the issue of whether the lines of the magnetic field are rotating or not, since the value of the vector $\mathbf{B}$ for a cylindrical magnet is the same whether or not the magnet is rotating.

A.17.8 Anticipation That $\nabla \cdot \mathbf{B} = 0$

Art. 3117 of [137] included the important conclusion that:
... there exist lines of force within the magnet, of the same nature as those without. What is more, they are exactly equal in amount to those without. They have a relation in direction to those without; and in fact are continuations of them, absolutely unchanged in their nature, so far as the experimental test can be applied to them. Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.

We recognize this as the first statement that $\nabla \cdot \mathbf{B} = 0$.

A.17.9 A Cylindrical Permanent Magnet is Equivalent to a Solenoidal Electromagnet

Art. 3120 of [137] remarked that the magnetic field of a uniformly magnetized cylinder is the same as that of a solenoid electromagnet of the same dimensions.

In 1832, Arts. 217-227 of [82], Faraday replaced the copper disk and the external magnet by a conducting, rotating magnet, whose self field acting on the "free" charges in the magnet also produced a current in the circuit, which effect is often called a homopolar generator.

A.17.10 Electric Lines of Force

The notion of electric lines of force, with tension along them and repulsion between them, appears in Art. 1297:
The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (1224.); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction (1304.).

His summary in Art. 1304 includes the statements:
I have used the phrases lines of inductive force and curved lines of force (1231. 1297. 1298. 1302.) in a general sense only, just as we speak of the lines of magnetic-force. The lines are imaginary, and the force in any part of them is of course the resultant of compound forces, every molecule being related to every other molecule in all directions by the tension and reaction of those which are contiguous.

A.17.11 The Magnetic Field

We have already noted that Faraday used the term *lines of magnetic force* in a footnote to Art. 114 of [81] (1831).

In 1845, Art. 2247 of [119], the term *magnetic field* appears for the first time in print:
The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.

A.17.12   Magnetic Power

In 1850, Art. 2806 of [136], Faraday wrote:

Any portion of space traversed by lines of magnetic power, may be taken as such a (magnetic) field, and there is probably no space without them. The condition of the field may vary in intensity of power, from place to place, either along the lines or across them...

2807. When a paramagnetic conductor, as for instance, a sphere of oxygen, is introduced into such a magnetic field, considered previously as free from matter, it will cause a concentration of the lines of force on and through it, so that the space occupied by it transmits more magnetic power than before (fig. 1). If, on the other hand, a sphere of diamagnetic matter be placed in a similar field, it will cause a divergence or opening out of the lines in the equatorial direction (fig. 2); and less magnetic power will be transmitted through the space it occupies than if it were away.

Fig. 1.   Fig. 2.
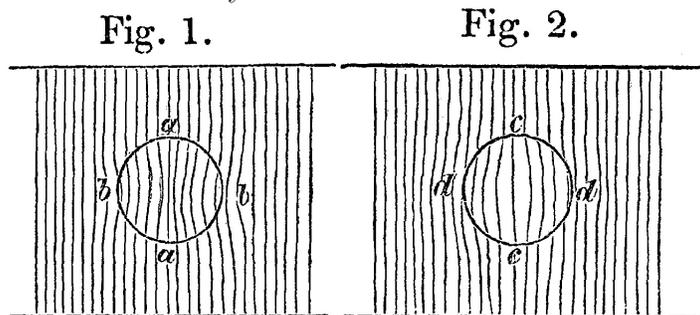
Here, one can identify Faraday's usage of the term *magnetic power* with the magnetic flux $\Phi_{\mathbf{B}} = \int \mathbf{B} \cdot d\mathbf{Area}$.

A further consequence of his interaction with Thomson appears to be that in 1852, beginning in sec. 3070 of [137], Faraday wrote about *lines of force* more abstractly, but without full commitment to their physical existence independent of matter. Thus, in Art. 3075 he stated:

I desire to restrict the meaning of the term line of force, so that it shall imply no more than the condition of the force in any given place, as to strength and direction; and not to include (at present) any idea of the nature of the physical cause of the phenomena...

A few sentences later he continued:

...for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the ether; for it is not at all unlikely that, if there be an ether, it should have other uses than simply the conveyance of radiations (2591. 2787.).

In Art. 3175, at the end of [137], he added:

...wherever the expression line of force is taken simply to represent the disposition of the forces, it shall have the fullness of that meaning; but that wherever it may seem to represent the idea of the physical mode of transmission of the force, it expresses in that respect the opinion to which I incline at present.

This has led many to infer that Faraday then believed in the physical existence of the lines of force even though he could not "prove" that.

Faraday's famous notion, that induced electrical currents are associated with wires "cutting" lines of magnetic force, is presented in Art. 3104, and a version of what is now called Faraday's law,

$$\mathcal{EMF} = -\frac{d}{dt}\int \mathbf{B} \cdot \text{Area},\tag{45}$$

is given verbally in Art. 3115,[80]
The quantity of electricity thrown into a current is directly as the amount of curves intersected.

In sec. 3117 Faraday noted that magnetic lines of force form closed circuits:
Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.
However, the last phrase indicates that Faraday did not have a clear view of what we call the strength of a magnetic field.

In sec. 3118 Faraday (re)affirmed that magnetic field lines do not rotate with a rotating magnet, and performs various experiments with what is now called a unipolar (or homopolar) generator to demonstrate this, which experiments are an early investigation of the relativity of rotating frames.

In 1852, Faraday also published a set of more speculative comments [139] in the Phil. Mag. (rather than Phil. Trans. Roy. Soc. London, the usual venue for his *Experimental Researches*), arguing more strongly for the physical reality of the lines of force.

In Art. 3258 he considered the effect of a magnet in vacuum, concluding (perhaps for the first time) that the lines of force have existence independent of a material medium:
A magnet placed in the middle of the best vacuum we can produce...acts as well upon a needle as if it were surrounded by air, water or glass; and therefore these lines exist in such a vacuum as well as where there is matter.

Faraday used examples of magnets and iron filings in various configurations to reinforce his vision of a tension along the lines of forces, and in sec. 3295 added the insight that there is a lateral repulsion between adjacent lines, referring to Fig. 5 below.

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[80]One should not infer from this that Faraday had an explicit notion of the magnetic field **B** as a measure of the density of lines of magnetic force. Rather, he emphasized the total number of lines within some area (the magnetic flux) as the *amount of magnetic force* (Art. 3109).
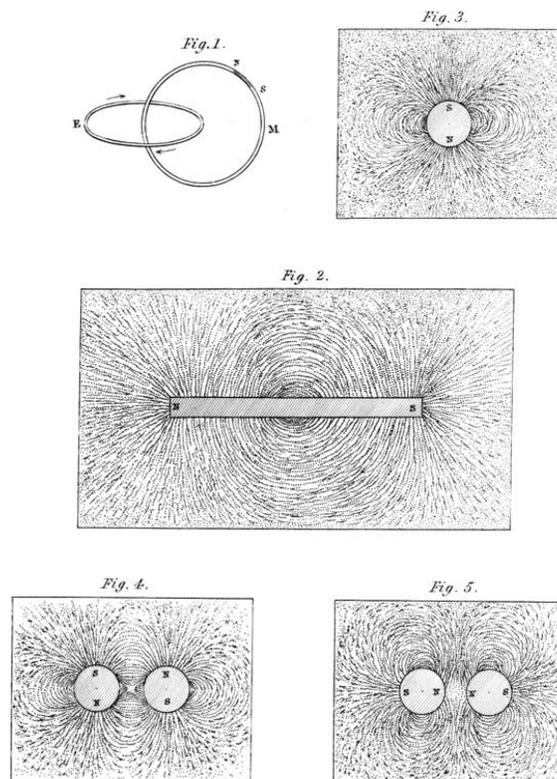
Faraday's last published comments on lines of force are in [153].

A.18   Henry

Henry began his studies of electromagnetism in 1827 [77], and in 1831 he demonstrated an electrical machine/motor of an unusual type [79].[81] In 1832 he was inspired by a brief report [85] of Faraday's discovery of electromagnetic induction to perform a series of experiments on mutual and self induction in circuits at rest [90, 97, 99, 104, 106].[82] Three illustrations (from [104]) of these experiments are shown below.

In 1840, sec. 56 of [106], Henry stated:

During the time a galvanic current is increasing in quantity in a conductor, it induces, or tends to induce, a current in an adjoining parallel conductor in an opposite direction to itself. In secs. 72-73 he suggested that Ohm's law could be applied to a secondary loop of resistance R in manner equivalent to $\mathcal{E}_{\text{induced}} = IR$, where the induced $\mathcal{EMF}$ is proportional to the rate of change of the current in the primary loop. We now call the proportionality constant the mutual inductance of the two loops. Thus, Henry gave the first sense of circuit analysis for circuits containing inductance.[83]

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[81]A replica of Henry's motor is discussed in [512].

See also https://www.princeton.edu/ssp/joseph-henry-project/

[82]A review of Henry's work on electromagnetic induction is given in [545]. Henry was the first to report an effect of self induction (1832), at the end of [90].

[83]A more explicit circuit analysis involving mutual inductance was given by Helmholtz (1847) in sec. VI8 of [124].

Fig. 1.　　Fig. 3.　　Fig. 4.

a represents coil No. 1, *b* helix No. 1, and *c, d*, handles for
receiving the shock.

a represents helix No. 4, *b* coil No. 1, in the form of a ring.
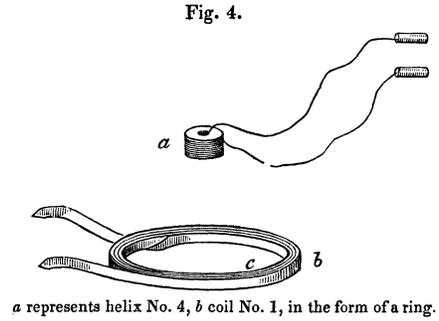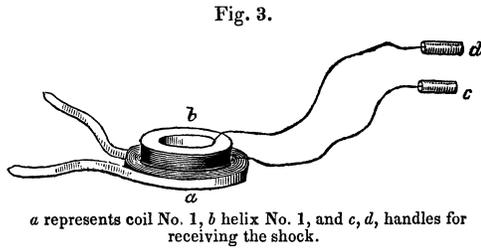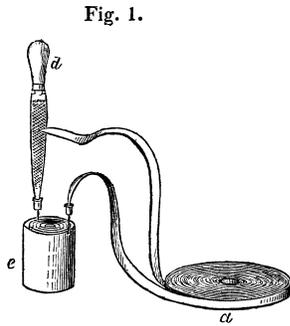
A.19　Lenz

In late 1833, Lenz [92] formulated his "law" (that effects of electromagnetic induction oppose
the effect which generates them) to characterize the direction of induced electromagnetic
currents.

A.20　Gauss

In 1867 Gauss posthumously published an analysis that he dated to 1835 (p. 609 of [98]), in
which he stated that a time-dependent electric current leads to an electric field which is the
time derivative of what we now called the vector potential. English translation from [437]:

> The Law of Induction
> Found out Jan. 23, 1835, at 7 a.m. before getting up.
> 1. The electricity producing power, which is caused in a point P by a current-element $\boldsymbol{\gamma}$, at a
> distance from P, $= r$, is during the time dt the difference in the values of $\boldsymbol{\gamma}/r$ corresponding
> to the moments t and dt, divided by dt. where $\boldsymbol{\gamma}$ is considered both with respect to size
> and direction. This can be expressed briefly and clearly by

$$ - \frac{\mathrm{d}(\boldsymbol{\gamma}/r)}{\mathrm{d}t} \, . \tag{46} $$

Gauss' unpublished insight that electromagnetic induction is related to the negative time
derivative of a scalar quantity was probably communicated in the late 1830's to his German
colleagues, of whom Weber was the closest.

On p. 612 (presumably also from 1835), Gauss noted a relation (here transcribed into
vector notation) between the vector $\mathbf{A} = \oint d\mathbf{l}/r$ and the magnetic scalar potential Ω of a
circuit with unit electrical current (which he related to the solid angle subtended by the
circuit on p. 611),

$$ - \boldsymbol{\nabla}\Omega = \boldsymbol{\nabla} \times \mathbf{A}. \tag{47} $$

While we would identify eq. (47) with the magnetic field $\mathbf{H}$, Gauss called it the "electricity-
generating force".

In any case, eq. (47) is the earliest appearance of the curl operator (although published
later than MacCullagh's use of this, p. 22 of [105] (1839)).

A.21   Grassmann

In 1845, Grassmann [112] remarked that although Ampère claimed [70] that his force law was *uniquement déduite de l'expérience*, it included the assumption that it obeyed Newton's third law. He noted that Ampère's law (33) implies that the force is zero for parallel current elements whose lie of centers makes angle $\cos^{-1}\sqrt{2/3}$ to the direction of the currents, which seemed implausible to him. Grassmann claimed that, unlike Ampère, he would make no "arbitrary" assumptions, but in effect he assumed that there is no magnetic force between collinear current elements, which leads to a force law,

$$\mathbf{F}_{\text{on } 1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on } 1}, \qquad d^2\mathbf{F}_{\text{on } 1} = \frac{\mu_0}{4\pi} I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{48}$$

in vector notation (which Grassmann did not use in [112], although he invented the notion of an exterior product of vectors in an n-dimensional space [111]). While $d^2\mathbf{F}_{\text{on } 1}$ is not equal and opposite to $d^2\mathbf{F}_{\text{on } 2}$, Grassmann showed that the total force on circuit 1 is equal and opposite to that on circuit 2, $\mathbf{F}_{\text{on } 1} = -\mathbf{F}_{\text{on } 2}$.

Grassmann's result is now called the Biot-Savart force law, eqs. (31)-(32),

$$\mathbf{F}_{\text{on } 1} = \oint_1 I_1 \, d\mathbf{l}_1 \times \mathbf{B}_2, \qquad \mathbf{B}_2 = \frac{\mu_0}{4\pi} \oint_2 \frac{I_2 \, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{49}$$

although Grassmann did not identify the quantity $\mathbf{B}_2$ as the magnetic field.

In 1844, Grassmann [111] invented linear algebra, which went largely unrecognized for 30 years. In 1877 [197], he related this to the algebra of Hamilton's quaternions [177], which together with a paper by Clifford (1878) [198], form the basis of contemporary geometric algebra.

A.22   Neumann

In 1845, Neumann [116] inferred from Lenz' law [92] that a conduction line segment $d\mathbf{s}$ that moves with velocity $\mathbf{v}$ near a magnet experiences a (scalar) electromotive force $d\mathcal{E}$ (with dimensions of energy) of the form (eq. (1), p. 15 of [116]),

$$d\mathcal{E} = E.Ds = -\epsilon v C.Ds, \tag{50}$$

where ϵ is a constant and C is a function of the magnet (and its geometric relation to the conducting line segment).

Although $\mathcal{E}$ is a scalar, Neumann had an intuition that the other quantities in his equation were not simply scalars but rather were what we now call vectors. For example, at the bottom of p. 66 he considered two line segments of lengths Ds and $D\sigma$ with components (Dx, Dy, Dz) and $D\xi, D\eta, D\zeta)$ with respect to rectangular axes, and mentioned the equality $DxD\xi + DyD\eta + DzD\zeta = \cos(D\sigma, Ds)D\sigma Ds$, we recognize as the scalar product $D\boldsymbol{\sigma} \cdot D\mathbf{s}$ of vectors $D\boldsymbol{\sigma}$ and $D\mathbf{s}$.

This has led many people to transcribe eq. (50) as,

$$d\mathcal{E} = \mathbf{E} \cdot Ds s = \mathbf{v} \times \mathbf{B} \cdot D\mathbf{s}, \tag{51}$$

and to credit Neumann with having been the first to identify the motional $\mathcal{EMF}$ of our eq. (6), although, like Ampère, Neumann had no concept of a magnetic (or electric) field.

On p. 67 of [116], Neumann discussed (what is now written as) the form (38), and verified that it gives the same total force between closed circuits as does Ampère's eq. (33).

Neumann did appreciate that electromagnetic induction occurs not only for moving conductors near a fixed magnet, but also for conductors at rest near a time-dependent magnet. This leads some people to credit him with also having stated the generalized flux law, our eq. (3).

Following the examples of Lagrange, Laplace and Poisson in relating forces of gravity and electrostatics to potentials, Neumann sought a potential for Ampère's force law (33) between two (closed) current loops. For this, he noted that this force law can be rewritten in the form (38), which permits us to write $\mathbf{F}_{\mathrm{on1}} = -\boldsymbol{\nabla}U$ where, U is the scalar potential (energy) given on p. 67 of [116],[84]

$$U = \frac{\mu_0}{4\pi}I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{52}$$

in SI units. We now also write this as,

$$U = I_i \oint_i d\mathbf{l}_i \cdot \mathbf{A}_j = I_i \int d\mathbf{Area}_i \cdot \boldsymbol{\nabla} \times \mathbf{A}_j = I_i \int d\mathbf{Area}_i \cdot \mathbf{B}_j = I_i \Phi_{ij}, \tag{53}$$

where Φ_{ij} is the magnetic flux through circuit i due to the current I_j in circuit j, and,

$$\mathbf{A}_j = \frac{\mu_0}{4\pi} \oint_j \frac{I_j \, d\mathbf{l}_j}{r} \,, \tag{54}$$

such that Neumann is often credited in inventing the vector potential $\mathbf{A}$, although he appears not to have written his eq. (52) in any of the forms of eq. (53). Yet, we can say that while Neumann had no concept of the magnetic field, he did emphasize a quantity with the significance of magnetic flux linked by a circuit.[85]

One application of his potential was given by Neumann in his eq. (4), p. 65 of [116]), which he expressed verbally on p. 68. This concerned the time integral (*Stromintegral*) of the current I_1 in one circuit, (with electrical resistance R_1, called $1/\epsilon'$ by Neumann), due to the motion of a second circuit whose current I_2 remains constant,

$$\int I_1 \, dt = I_2 \frac{V_i - V_f}{R_1} \,, \qquad \text{where} \qquad V = \frac{\mu_0}{4\pi} \oint_i \oint_j \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{56}$$

and $V_{i,f}$ are the initial and final potentials for unit currents (*i.e.*, the initial and final mutual inductances).[86] This result has the implication that the $\mathcal{EMF}$ induced in circuit 1 due to

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[84]If we write eq. (52) as $U = I_1 I_2 M_{12}$, then M_{12} is the mutual inductance of circuits 1 and 2.

[85]Note also that using our eq. (36), due to Ampère, the magnetic flux can be written as,

$$\Phi_{ij} = \frac{\mu_0 I_j}{4\pi} \oint_i \oint_j \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{r} = -\frac{\mu_0 I_j}{4\pi} \oint_i \oint_j \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \frac{dl_i \, dl_j}{r} = -\frac{\mu_0 I_j}{4\pi} \oint_i \oint_j \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})}{r} \,, \tag{55}$$

since the integral of $\partial^2 r / \partial l_i \partial l_j$ around a closed loop vanishes, and the third form follows from the second recalling eq. (35).

[86]Equation (50) is sometimes called Felici's law.

the motion of circuit 2 has the form,

$$\mathcal{E} = -I_2 \frac{dV}{dt} = -\frac{d}{dt}(I_2 V) = -\frac{d\Phi_{1,2}}{dt}, \tag{57}$$

where $\Phi_{1,2}$ is the magnetic flux linked by circuit 1 due to the current in circuit 2. This was not explicitly stated by Neumann in [116], but apparently many astute readers did infer that the induced $\mathcal{EMF}$ is the negative time derivative of an appropriate version of Neumann's potential. Hence, Neumann is often credited with formulating an early version of what we now call Faraday's law.

A.23 Weber (Translated works in [560]-[563])

The term unipolar induction for Faraday's homopolar dynamo is due to Weber (1839) [107].

Weber was perhaps the last major physicist who did not use electric and magnetic fields to describe electromagnetism, postulating instead an (instantaneous) action-at-a-distance formulation for the (central) force between charges (1846, p. 375 of [121], p. 144 of [251]),[87]

$$\mathbf{F}_{\text{Weber}} = -\frac{ee'}{r^2} \left[1 - \frac{a^2}{16} \left(\frac{\partial r}{\partial t} \right)^2 + \frac{a^2 r}{8} \frac{\partial^2 r}{\partial t^2} \right] \hat{\mathbf{r}} \tag{58}$$

where $\mathbf{r} = \mathbf{r}_{e'} - \mathbf{r}_e$. This was the first published force law for moving charges (which topic Ampère refused to speculate upon). The constant a has dimensions of velocity^{-1}, and was later (1856) written by Weber and Kohlrausch, p. 20 of [161], as $4/C$, who noted that their C is the ratio of the magnetic units to electrical units in the description of static phenomenon, which they determined experimentally to have a value close to 4.4×10^8 m/s. Apparently, they regarded it as a coincidence that their C is roughly $\sqrt{2}$ times the speed c of light.[88]

In 1848, p. 229 of [126], Weber related his force law (58) to the (velocity-dependent) potential,[89,90]

$$U_{\text{Weber}} = \frac{ee'}{r} \left[1 - \frac{a^2}{16} \left(\frac{dr}{dt} \right)^2 \right]. \tag{60}$$

Weber showed that his eq. (58) can be used to deduce Ampère's force law, as well as Neumann's potential (52) (as also discussed by the latter in sec. 5 of his 1847 paper [125]).[91]

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[87]For an extensive discussion of Weber's electrodynamics, see [447]. Maxwell gave a review of the German school of electrodynamics of the mid 19th century in the final chapter 23 of his *Treatise* [248].

[88]See [499] for the history of the use of c to represent the speed of light.

[89]Weber had in effect deduced this potential in 1846, p. 375 of [121], but did not identify it as such.

[90]Weber's force law (58) has the form $\mathbf{F} = f(r,t), \hat{\mathbf{r}}$, where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ with $\mathbf{x}(\mathbf{x}')$ being the positions of charges $e(e')$. If the force can be deduced from a potential U according to $\mathbf{F} = -\boldsymbol{\nabla}U$, we expect that U is also a function only of r and t. In this case $[\boldsymbol{\nabla}U]_x = \partial V/\partial x = (\partial U/\partial r)\partial r/\partial x = (\partial U/\partial r)(x - x')/r$. That is, $-\boldsymbol{\nabla}U = -(\partial U/\partial r)\hat{\mathbf{r}}$. For Weber's potential (60), we have,

$$-\frac{\partial U_{\text{Weber}}}{\partial r} = \frac{ee'}{r^2} \left[1 - \frac{a^2}{16} \left(\frac{\partial r}{\partial t} \right)^2 + \frac{a^2}{8} r \frac{\partial r}{\partial t} \frac{\partial}{\partial r} \frac{\partial r}{\partial t} \right] \quad \text{and} \quad \frac{\partial}{\partial r} \frac{\partial r}{\partial t} = \frac{\partial t}{\partial r} \frac{\partial^2 r}{\partial t^2}, \tag{59}$$

so indeed $\mathbf{F}_{\text{Weber}} = -\boldsymbol{\nabla}U_{\text{Weber}}$.

[91]A comparison of the theories of Neumann and Weber is given in sec. 2 of [444], which also includes a review of Helmholtz' later theories [193].

In secs. 28 and 30 of [121], Weber consider the induction in a current element $d\mathbf{l}'$ due to a variable current I in element $d\mathbf{l}$, with both elements at rest, and deduced that,

$$d^2\mathcal{E}' = -\frac{\mu_0}{4\pi}\frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})}{r}\frac{dI}{dt}\,, \quad \text{whose integral is} \quad \mathcal{E}' = -\frac{\mu_0}{4\pi}\oint\oint\frac{d\mathbf{l}\cdot d\mathbf{l}'}{r}\frac{dI}{dt} = -M\frac{dI}{dt}\,, (61)$$

recalling eq. (55), and where M is the mutual inductance in the case the elements $d\mathbf{l}$ and $d\mathbf{l}'$ are in different circuits. Weber seemed little concerned with circuit analysis, did not mention integral form of eq. (61), and downplayed the merits of Neumann's "potential" compared to his own, eq. (60). However, it may be the Weber was the first to explicitly note that induced $\mathcal{EMF}$'s are proportional to a time derivative (although this result is implicit in Faraday's discussion, starting with that in [81]).

Weber's electrodynamics was more ambitious than that of Neumann's (which was tacitly restricted to quasistatic examples with low-velocity charges), as Weber sought to describe charges with arbitrary velocities and accelerations. Neumann's contributions, within their realm of applicability have aged well, while Weber's electrodynamics is largely forgotten as it does not contain electromagnetic radiation.

Weber's most lasting contribution to electrodynamics was his theory of paramagnetism [127, 141], developed following Faraday's studies of diamagnetism [119, 120] and paramagnetism [135] (which terms were coined by Faraday). Weber considered that paramagnetic "atoms" are objects with a permanent electric current circulating around a diameter.[92]

A.24 Kirchhoff

A.24.1 Circuit Laws

In 1845, Kirchhoff (age 21, and student of Neumann) gave his circuit laws [115, 122] for a network of batteries and resistors, as a generalization of Ohm's law [76] that a circuit consisting of a battery with electromotive force $\mathcal{E}$ and electrical resistance R supports and electric current I related by $\mathcal{E} = IR$. For example, in a circuit consisting of a single loop with several batteries and several resistors, the circuit (loop) equation is $\sum\mathcal{E} = I\sum R$.

Kirchhoff never applied his laws to time-dependent circuits.

In 1849, Kirchhoff [129] noted that the $\mathcal{EMF}$ in Ohm's law for steady currents in circuits consisting only of batteries and resistors is the difference between the electric scalar potential $V = \int \rho\, d\text{Vol}/4\pi\epsilon_0\, r$ at two relevant points.

A.24.2 Waves of Electric Potential

In 1857, Kirchhoff published two papers [163, 164] on the motion of electricity in wires, based on Weber's (action-at-a-distance) electrodynamics [121]. The consideration of a single wire by Kirchhoff followed the typical practice of telegraphy at the time [140, 215], in which only a single wire appeared to be involved, and the "ground"/"earth" served as a return path to

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[92]This view was discussed by Faraday (1854) [149], and by Maxwell in Art. 843 of [191].

"complete the circuit". As such, neither the capacitance between the wire and "ground" nor the self inductance of the wire + "ground" circuit were considered.[93]

Weber followed Neumann (1845) [116] who had introduced the concept of mutual inductance M of two circuits (but not the self inductance L of a single circuit), as well as the vector potential $\mathbf{A}$. For example, the magnetic flux Φ_{12} through circuit 1 due to current I_2 in circuit 2 is related by $\Phi_{12} = M_{12}I_2 = \int \mathbf{B}_2 \cdot d\mathbf{Area}_1 = \oint \mathbf{A}_2 \cdot d\mathbf{l}_1$. We would add that the integral form of Faraday's law then tells us that if current I_2 varies with time, a (scalar) $\mathcal{EMF}$ is induced in circuit 1 related by $\mathcal{E}_{12} = -\dot{\Phi}_{12} = -M_{12}\dot{I}_2$. Since Neumann, Weber and Kirchhoff had no concept of magnetic flux they did not say this, but rather emphasized the (vector) electromotive force (*electromotorische Kraft*) of one current element on another, and that the (scalar) $\mathcal{EMF}$ on current element 1 due to changes in current element 2 as $\mathcal{E}_{12} = -k_{12}\dot{I}_2$, where k_{12} is a geometric factor. See the top of p. 199 of [163], where the part of W due to $i' = I_2$ is proportional to $M_{12}I_2$.

Rather impressively, Kirchhoff [163] deduced a wave equation for the current and charge on current elements (conductors) of small resistance, finding the wavespeed to be $c = 1/\sqrt{\epsilon_0\mu_0}$, where the constants ϵ_0 and μ_0 can be determined from electro- and magneto-static experiments (Weber and Kohlsrausch (1856) [161]), and the value of c was close to the speed of light as then known. However, as Weber's electrodynamics was based on action-at-a distance, and was not a field theory in the sense of Faraday, Weber and Kirchhoff did not infer that, since electric waves on wire moved at light speed, light must be an electromagnetic phenomenon.

We now consider that electromagnetic waves associated with conductors are almost entirely outside the conductors, and that the wavespeed of surface charge and current densities matches the wave speed in the medium outside the conductor, *i.e.*, c in case of vacuum. Kirchhoff's argument was the first demonstration of the latter result, which holds for waves propagating parallel to the surface of a conductor of "any" shape. In this sense, Kirchhoff did not deduce the "telegrapher's equation" (due to Heaviside (1876) [195]) for transmission lines based on two, parallel conductors, for which the wave speed is $v = 1/\sqrt{LC}$, where L and C are the inductance and capacitance per unit length. This behavior is consistent with Kirchhoff's result because of a general (geometrical) "theorem" that $LC = \epsilon_0\mu_0 = 1/c^2$ for a one-dimensional transmission line, if dielectric effects can be ignored and the current flows only on the surface of the conductors. See, for example, [467].

In [163], Kirchhoff discussed waves on a straight wire, and a circular wire loop of circumference $= n\lambda$ (which now finds application as a self-resonant loop antenna, particularly for $n = 1$; see, for example, sec. 2.2.2 of [510]). An ingredient in his analysis was that the electromagnetic potentials V and $\mathbf{A}$ were subject to the condition $\boldsymbol{\nabla} \cdot \mathbf{A} = (1/c)\partial V/\partial t$ (in Gaussian units), now known as the Kirchhoff gauge [497], the first-ever use of a gauge condition. *In Maxwell's theory and in the Kirchhoff gauge, the potential V can be said to propagate with imaginary velocity, ic [497].*

Kirchhoff, eq. (2), p. 199 of [163], expressed Ohm's law [76] in a form equivalent to $\mathbf{J} = \sigma(-\boldsymbol{\nabla}\varphi - \partial\mathbf{A}/\partial ct)$ $[= \sigma\mathbf{E}]$, where σ is the electrical conductivity, calling $\mathbf{E}$ the *elektro-*

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[93]The kind of derivation of the "telegrapher's equation" for a two-wire telegraph "line" found in textbooks today, using Kirchhoff's circuit laws, was first given by Heaviside in 1876 [195]. See also the Appendix of [460].

motorische Kraft (electromotive force).

Kirchhoff also gave an analysis [164] for a loop of wire of significant resistance, similar to that of Thomson [156], that the potential (and electric charge density) on a wire of length l with resistance R and capacitance C obeys the diffusion equation $d^2V/dx^2 = (RC/l^2)dV/dx$ (which does not involve the constant c), when effects of self inductance are ignored. *It may be that contemporary readers of Kirchhoff's analysis emphasized the case of agreement with Thomson's diffusion model, rather than propagation at the speed of light.*[94]

A.24.3 Analysis of an *L-R-C* Circuit

In 1864, Kirchhoff analyzed a series *L-R-C* circuit [174] without reference to his circuit laws, but including the $\mathcal{EMF}$'s associated with the inductor and capacitor in the manner of present circuit analysis. He cited Thomson's analysis (1853) [145] of this circuit, but did not use Thomson's energy argument (Appendix A.2).

A.25 Helmholtz

A.25.1 Conservation of Energy

In 1847, Helmholtz published an important memoir on conservation of "force" *Kraft* [124],[95] where "force" means energy. Section V discussed electrostatics, steady currents and thermoelectric effects, including emphasis that I^2R represents the electrical energy dissipated as Joule heating by an electric current I (Helmholtz' J) in an electrical resistance R (Helmholtz' W).[96]

Section VI discussed electromagnetism, with commentaries on the then-recent works of Lenz, Neumann and Weber.

In sec. VI5, Helmholtz stated that motion of a magnet near an electric circuit induces an electromotive force $-dV/dt$ where V is the "potential towards the current" (*Potential gegen den von der Stromeinheit durchlaufenen Leiter*). If we regard V as the magnetic flux Φ of the magnet that is linked by the circuit, then this is perhaps the first explicit statement of the generalized flux law (3).

Helmholtz added that the charge which flows in the circuit is related to the difference between the initial and final "potentials",

$$\Delta Q = \int I\, dt = \frac{V_i - V_f}{R}\,, \tag{62}$$

where R is the electrical resistance of the circuit. Helmholtz attributed this result to Neumann [116].

In sec. VI8 of [124], Helmholtz considered the effect of motion of one circuit with respect to another, writing their magnetic interaction energy as $I_1 I_2 V$, where here V is the "potential for

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[94]See, for example, remarks by Weber (1863) on p. 6 of [172].

[95]Helmholtz' paper was reviewed by Maxwell in Arts. 543-544 of [248].

[96]Joule (1841) [108] had established that a current I through a resistor R generates heat at the rate I^2R whether or not I is constant, as an example of conservation of energy. This followed the much earlier demonstration of the mechanical equivalent of heat by Rumford [18].

unit currents" (*Potential für die Stromeinheit gegen einander*), *i.e.*, the mutual inductance. He supposed that circuit 1 was in motion, but with a steady current I_1, and deduced that the current I_2 in circuit 2 (with electrical resistance R_2 and a battery of $\mathcal{EMF}$ A_2 is given by,

$$I_2 = \frac{A_2 - I_2\, dV/dt}{R_2}\,. \tag{63}$$

He claimed that this result had been shown experimentally by Weber in [121] (but I have not yet found it there).

In an Addendum written in 1881, p. 55 of [238], Helmholtz noted that a velocity-dependent force of the form $\mathbf{v} \times \mathbf{B}$, where $\mathbf{B}$ is a vector, does not change the kinetic energy of a particle of velocity $\mathbf{v}$. He did not relate $\mathbf{B}$ to the magnetic field.

A.25.2 Circuit Analysis

In 1850, Helmholtz studied the propagation of electrical signals in nerves, modeling this as a resistor and a coil/inductor. When a sharp rise in electric potential was applied to this circuit, the current took a characteristic time to approach its asymptotic value. Helmholtz reported experimental results in [131, 132], and gave a circuit analysis in [134]. He did not derive a circuit equation, but simply stated a hypothesis (*Voraussetzung*), p. 510 of [134], to be confirmed (or not) by experiment, that Ohm's law would be modified by the presence of the coil to read, Helmholtz' eqs. (2)-(3),

$$IR = \mathcal{E} - L\frac{dI}{dt}\,, \qquad I = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right), \tag{64}$$

where I is the current, which is zero at time $t = 0$ when electromotive force $\mathcal{E}$ is applied to the circuit that has resistance R and self-inductance L.[97]

Another contribution to DC circuit theory was made by Helmholtz in 1853 [143], when he discussed what is now called Thévinin's theorem [93], that any complicated but "linear" circuit is equivalent to a single source of electromotive force and a single resistance (or impedance in case of time-dependent circuits,[98,99]

A.25.3 Helmholtz' Theorem

In 1858, Helmholtz [166] deduced that "any" vector field $\mathbf{E}$ can be related to a scalar potential Ψ and a vector potential $\mathbf{A}$ as $\mathbf{E} = \boldsymbol{\nabla}\Psi + \boldsymbol{\nabla} \times \mathbf{A}$, where $\Psi = 0$ if $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ and $\mathbf{A} = 0$ if $\boldsymbol{\nabla} \times \mathbf{E} = 0$. See also [504].

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[97]Helmholtz called the self inductance the *Potential P*, following Neumann (1845) [116], who apparently was consulted by Helmholtz during these studies.

[98]The term impedance was introduced by Heaviside in 1886 [228].

[99]For historical comments on Helmholtz, Thévinin and others, see [488].

A.25.4 Electrodynamics

Helmholtz later discussed a theory of electrodynamics that was a kind of hybrid of Maxwell's theory and that of Neumann and Weber, starting with [183] (1870).[100] See also [185, 187, 192, 193, 206], and commentaries in [383, 441, 444, 449].

In eq. (1), p. 76 of [183], Helmholtz noted that a general form for the magnetic interaction energy (his P, but our U) of two current elements could be written as a combination of forms he attributed to Neumann and Weber,[101]

$$d^2U = \frac{\mu_0}{4\pi}\left(\frac{1+k}{2}\frac{I_1\,d\mathbf{l}_1 \cdot I_2\,d\mathbf{l}_2}{r} + \frac{1-k}{2}\frac{(I_1\,d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(I_2\,d\mathbf{l}_2 \cdot \hat{\mathbf{r}})}{r}\right), \tag{65}$$

where $k = 1$ for Neumann's form and $k = -1$ for Weber's. Then in eq. (1^a) he noted that the scalar U is related to a vector potential (his (U, V, W) but our $\mathbf{A}$) as $U = \int \mathbf{J}\cdot\mathbf{A}\,d\text{Vol}/2$, noting that $I\,d\mathbf{l} \leftrightarrow \mathbf{J}\,d\text{Vol}$ where $\mathbf{J}$ is the current density, and the vector potential is,

$$\mathbf{A} = \frac{1+k}{2}\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}}{r}\,d\text{Vol} + \frac{1-k}{2}\frac{\mu_0}{4\pi}\int\frac{(\mathbf{J}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r}\,d\text{Vol} = \frac{1+k}{2}\mathbf{A}_\text{N} + \frac{1-k}{2}\mathbf{A}_\text{W}, \tag{66}$$

although Neumann never wrote the form called $\mathbf{A}_\text{N}$ here. Kirchhoff, p. 530 of [164], attributed $\mathbf{A}_\text{W}$ to Weber, who later transcribed Kirchhoff's paper into sec. I.1. of [173], with $\mathbf{A}_\text{W}$ appearing on p. 578.[102] Helmholtz' discussion was tacitly restricted to magnetostatics, such that his eq. (3^a), p. 80, that $\nabla \cdot \mathbf{A} = k\,dV/dt$, where V is the instantaneous electric scalar potential, led him to identify $k = 0$ with Maxwell's theory [176], with its emphasis on $\nabla \cdot \mathbf{A} = 0$.

A.26 W. Thomson (Lord Kelvin)

A.26.1 Force Fields

In 1842 (at age 18!), W. Thomson [110] noted an analogy between the (vector) flow of heat and the "attractive force" of electricity. At that time he was concerned with electrostatics, for which it is natural to consider the force only at the locations of charges and not in the space between them. In contrast, the flow of heat exists in the space between sources and sinks of heat, so Thomson's analogy perhaps started him thinking about possible significance of electrical forces away from the location of electric charges.

Thomson appears to have become aware of Faraday's work in 1845,[103] and soon published transcriptions [117] of some of Faraday's concepts into mathematical form. He noted the contrast between Coulomb's action-at-distance view of electrical forces, and Faraday's view (reminiscent of Descarte's) that these forces are transmitted via some kind of "action of contiguous particles of some intervening medium", and proceeded to argue that these are

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[100]For comments by the author on this paper, see [549, 558].

[101]See also sec. IIB of [475], and [537]. The energy that Helmholtz associated with Weber was never actually advocated by the latter, who had a different vision of magnetic energy, as discussed above in A.23.

[102]Note that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is the same for $\mathbf{A}_\text{N}$ and $\mathbf{A}_\text{W}$, which provides an early example of gauge invariance.

[103]For a review of Thomson's efforts on electrostatics in 1841-45, see [404].

what might now be called "dual" explanations of electricity. We see in this discussion the beginning of Thomson's lifelong vision of a mechanical ether supporting electricity and magnetism.[104]

In 1846, Thomson [123], p. 63, described the electrical force due to a unit charge at the origin "exerted at the point (x, y, z)" as $\mathbf{r}/r^3$, without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term "electric field" were used by Thomson.

He immediately continued with the example of a "point" magnetic dipole $\mathbf{m}$, whose scalar potential is $\Phi = \mathbf{m} \cdot \mathbf{r}/r^3$, noting that the magnetic force $-\boldsymbol{\nabla}\Phi$ on a unit magnetic pole p can also be written as $\boldsymbol{\nabla} \times \mathbf{A}$ where $\mathbf{A} = \mathbf{m} \times \mathbf{r}/r^3$ (although Thomson did not assign a symbol to the vector $\mathbf{A}$). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [116] implied this result, but was not explicit about its application to Thomson's example).

In a major paper on magnetism in 1849 [128], Thomson still did not use the term "field", but wrote in sec. 48:
The resultant force at a point in space, void of magnetized matter, is the force that the north pole of a unit-bar (or a positive unit of imaginary magnetic matter), if placed at this point, would experience.

The term "magnetic field" in the contemporary sense first appears in 1851 on p. 179 of [133], where Thomson wrote:
Definition.—Any space at every point of which there is a finite magnetic force is called "a field of magnetic force;" or, magnetic being understood, simply "a field of force;" or, sometimes, "a magnetic field."
Definition.—A "line of force" is a line drawn through a magnetic field in the direction of the force at each point through which it passes; or a line touched at each point of itself by the direction of the magnetic force.

A.26.2   B and H

In secs. 74-75 of [128] (1851), Thomson noted that a static magnetic field can have zero divergence (*solenoidal*) or zero curl (*lamellar*).[105] In sec. 78 he introduced the magnetic-field vector (X, Y, Z) that we now identify with $\mathbf{H}$, while discussing the effect of the magnetic field on hypothetical magnetic poles inside small cavities in a medium with magnetization density $\mathbf{M}$. However, he did not make a connection between a *solenoidal* magnetic field and the magnetic field of the form $\boldsymbol{\nabla} \times \mathbf{A}$ that he had discussed in [123].

In 1856, Maxwell identified two magnetic fields, now called $\mathbf{B}$ and $\mathbf{H}$, on p. 54 of [158] without explicit reference to Thomson. Maxwell considered the two fields to be related by $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability.

It seems that in 1871, Thomson was the first to relate the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ as

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[104]*I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand; and that is why I cannot get the electro-magnetic theory.* Page 603 of [219].

[105]Helmholtz' theorem, sec. A.25.3 above, was still several years in the future.

$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ ($\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$ in SI units), in eq. (r), p. 401 of [217].[106,107,108]

A.26.3   The Telegraph

In 1855, Thomson [156] published an analysis of the telegraph based on its electric resistance and capacitance per unit length, and inferred that the electric potential V propagates in a manner analogous to heat conduction.

A.27   Felici

In 1854, Felici published extensive studies [151] of electromagnetic induction, building on work of Neumann and Weber,[109] which included the statement in sec. 62 that the time integral of the electromotive force induced in circuit 1 by circuit 2 is related to the difference between a certain quantity at the initial and final times,

$$\int_{t_1}^{t_2} \mathcal{E}_1 \, dt = \frac{\mu_0 I_2}{4\pi} \oint_1 \oint_2 \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \frac{dl_1 \, dl_2}{r}\bigg|_{t_1} - \frac{\mu_0 I_2}{4\pi} \oint_1 \oint_2 \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \frac{dl_1 \, dl_2}{r}\bigg|_{t_2} \tag{67}$$

Recalling our eq. (55), we recognize this quantity as Φ_{12}, such that Felici's Law (67) is the time integral of the generalized flux law (3),

$$\int_{t_1}^{t_2} \mathcal{E}_1 \, dt = -\int_{t_1}^{t_2} \frac{d\Phi_{1,2}}{dt} \, dt = \Phi_{12}(t_1) - \Phi_{12}(t_2). \tag{68}$$

This result is closely related to our eq. (50), due to Neumann, discussed in sec. A.22 above.[110]

A.28   Maxwell

Maxwell published his developments of the theory of electrodynamics in four steps, *On Faraday's Lines of Force* [158] (1856), *On Physical Lines of Force* [167, 168, 169, 170] (1861-

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[106]In eqs. (k)-(l), p. 399 of [217] (1871), Thomson stated that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, and $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi\mathbf{J}/c$. These relations were previously given by Maxwell (1856), pp. 54-56 of [158]. See aslo secs. A.28.1.3-4 below.

[107]Complexities of the distinction between $\mathbf{B}$ and $\mathbf{H}$ are reviewed in [469].

[108]Neither Thomson nor Maxwell enunciated a concept of the polarization density $\mathbf{P}$ of electric dipoles, and only regarded the relation between $\mathbf{D}$ and $\mathbf{E}$ as $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is now called the (relative) dielectric constant and/or the (relative) permittivity. See Art. 111 of [247] for Maxwell's use of the term polarization.

In 1885, Heaviside introduced the concept of an electret as the electrical analog of a permanent magnet [221], and proposed that the electrical analog of magnetization (density) be called electrization. He did not propose a symbol for this, nor did he write an equation such as $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$.

The density of electric dipoles was called the polarization by Lorentz (1892) in sec. 102, p. 465 of [246], and assigned the symbol $\mathbf{M}$.

Larmor (1895), p. 738 of [222], introduced the vector (f', g', h') for what is now written as the polarization density $\mathbf{P}$, and related it to the electric field $\mathbf{E} = (P, Q, R)$ as $(f', g', h') = (K - 1)(P, Q, R)/4\pi$, *i.e.*, $\mathbf{P} = (\epsilon - 1)\mathbf{E}/4\pi = (\mathbf{D} - \mathbf{E})/4\pi$. Larmor's notation was mentioned briefly on p. 91 of [261] (1898).

The symbol $\mathbf{M}$ for dielectric polarization was changed to $\mathbf{P}$ by Lorentz on p. 263 of [265] (1902), and a relation equivalent to $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ was given in eq. (22), p. 265. See also p. 224, and eq. (147), p. 240 of [266] (1903), which latter subsequently appeared as eq. (142), p. 155 of the textbook [268] (1904) by Abraham.

[109]A summary of Felici's work appeared on pp. 719-722 of Vol. III of de la Rive's *Treatise* (1858) [165].

[110]Felici further emphasized his law in [159].

62), *A Dynamical Theory of the Electromagnetic Field* [176] (1864), and in his *Treatise on Electricity and Magnetism* [190, 191, 247, 248] (1873).

As Appendix A.28 is rather long, we first preview Maxwell's arguments most directly related to Faraday's Law, and to the force on a moving charge.[111]

Appendix A.28.1.7 notes Maxwell's first discussion of a *moving* (charged) *particle* on p. 64 of [158] (1856), in which he took the convective derivative of the vector potential to obtain our eq. (79), whose meaning was not very apparent. It it interesting to this author that Maxwell did not, in 1856 or later, convert our eq. (79) into our (81), which was subsequently claimed by Helmholtz [193], Larmor [214], Watson [236], and J.J. Thomson, p. 260 of [248], to have been what Maxwell should have done.

Appendix A.28.1.8 discusses Maxwell's first (verbal) statement of Faraday's law, p. 50 of [158] (1856).

Appendix A.28.2.3 presents Maxwell's deduction of Faraday's law via an energy argument, Prop. VII, pp. 288-291 of [168] (1861), which included the first statement of the differential form of the law.

Appendix A.28.2.6 reviews Maxwell's use (1861) [168] of his theory of molecular vortices and an energy argument to deduce our eq. (98), which is the curl of the Lorentz force law. Maxwell then integrated this by adding the term $\nabla\Psi$ to the argument of the curl operator, which yields the "Lorentz" force law, our eq. (99), if we accept Maxwell's interpretation of Ψ as the *electric tension* (electric scalar potential).

Appendix A.28.2.7 reports additional discussion of Faraday's law by Maxwell in 1861, [168], when moving conductors are involved.

Appendix A.28.3.1 considers Maxwell's discussion of Faraday's law in secs. 24 and 50 of [176] (1864).

Appendix A.28.3.6 discusses Maxwell's consideration (1864), sec. 63 of [176], of a circuit at rest, which led him to identify the *electromotive force* vector on a charge at rest as our eq. (124).

Appendix A.28.3.7 recounts Maxwell's extrapolation in sec. 64 of [176] to a moving charge (circuit element) by the addition of the single-charge version of the Biot-Savart force law, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, to arrive at the "Lorentz" force law, our eq. (129), in the same form, our eq. (99), as he had previously found in [168].

Appendix A.28.4.2 discusses Maxwell's first mention of Faraday's law in his *Treatise* [191], Arts. 489-490, where only the force on a moving circuit element was considered. Maxwell's verbal statement of Faraday's law, in Arts. 530 and 541, is reviewed in secs. A.28.4.4-5.

The "Lorentz" force law, our eqs. (99) and (129), was also deduced by Maxwell (1873) in Arts. 598-599 of [191] via a slightly different argument, as discussed in Appendix A.28.4.6.

Maxwell included mention of this force law in Art. 619 of [191], the summary of his theory of the electromagnetic field, but in a somewhat unfortunate manner, as reviewed in Appendix A.28.4.9.

A.28.1 In *On Faraday's Lines of Force* [158]

For a discussion of Maxwell's thoughts in 1855, which culminated in the publication of [158], see [454].

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[111]Another overview of Maxwell's efforts on electromagnetism is [529].

A.28.1.1 *Theory of the Conduction of Current Electricity* and *On Electromotive Forces*

On p. 46 of [158], Maxwell stated: *According to the received opinions we have here a current of fluid moving uniformly in conducting circuits, which oppose a resistance to the current which has to be overcome by the application of an electro-motive force at some part of the circuit.*

He continued on pp. 46-47: *When a uniform current exists in a closed circuit it is evident that some other forces must act on the fluid besides the pressures. For if the current were due to difference of pressures, then it would flow from the point of greatest pressure in both directions to the point of least pressure, whereas in reality it circulates in one direction constantly. We must therefore admit the existence of certain forces capable of keeping up a constant current in a closed circuit. Of these the most remarkable is that which is produced by chemical action. A cell of a voltaic battery, or rather the surface of separation of the fluid of the cell and the zinc, is the seat of an electro-motive force which can maintain a current in opposition to the resistance of the circuit. If we adopt the usual convention in speaking of electric currents, the positive current is from the fluid through the platinum, the conducting circuit, and the zinc, back to the fluid again.*

Here, Maxwell seemed to accept the *received opinions*[112] that electric current is a fluid; and actually two counterpropagating fluids.

A.28.1.2 Ohm's Law, Electromotive Force and the Electric Field

On p. 47 of [158], Maxwell wrote a version of Ohm's Law as $F = IK$ for an electrical circuit of resistance K that carries current I driven by a battery. He calls F the *electromotive force*, which is consistent with a more contemporary notation $\mathcal{E}$ (with dimensions of electric potential (voltage) rather than of force).[113]

On p. 53, Maxwell introduced the (free/conduction) electric current-density vector $\mathbf{J} = (a_2, b_2, c_2)$, the electric scalar potential $\Psi = p_2$, and the electric field $\mathbf{E} = (\alpha_2, \beta_2, \gamma_2)$, writing in his eq. (A),
$$\mathbf{E} = \mathbf{E}_{\text{other}} - \boldsymbol{\nabla}\Psi \qquad \left[\alpha_2 = X_2 - \frac{dp_2}{dx}, \ etc.\right], \tag{69}$$

with $\mathbf{E}_{\text{other}} = (X_2, Y_2, Z_2)$ being a possible contribution to the electric field not associated with a scalar potential. To possible confusion, Maxwell called the vector $\mathbf{E}$ an *electro-motive force*, which term he also used for the scalar $\mathcal{E}$.

Also on p. 53, Maxwell introduced the electrical resistivity $\varrho = k_2$ (reciprocal of the electrical conductivity $\sigma = 1/\varrho$), so that Ohm's Law can be written as Maxwell's eq. (B),
$$\mathbf{E} = \varrho\,\mathbf{J}_{\text{free}} = \frac{\mathbf{J}_{\text{free}}}{\sigma} \qquad [\alpha_2 = k_2\,a_2, \ etc.], \tag{70}$$

On p. 54, Maxwell noted that for any closed curve eq. (69) implies,
$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{E}_{\text{other}} \cdot d\mathbf{l}, \tag{71}$$

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[112]Maxwell cited French translations of papers by Kirchhoff [129] and Quincke [160]. He seemed unaware that Kirchhoff had also published an English version of his paper [129].

[113]On p. 47 of [158], Maxwell also developed a magnetic-circuit analog of Ohm's law. See, for example, p. 97 of [409].

He also introduced the concept of the flux (*conduction*) $\mathbf{E} \cdot d\mathbf{S}$ of a field $\mathbf{E}$ across a surface element $d\mathbf{S}$, and noted (Gauss' Law) that for a closed surface,

$$\int \mathbf{E} \cdot d\mathbf{S} = \int \boldsymbol{\nabla} \cdot \mathbf{E} \, d\text{Vol} = 4\pi \int \rho \, d\text{Vol}, \qquad \text{with} \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho. \tag{72}$$

He indicated in his eq. (C), p. 54, that he will often write the divergence of a vector field as $4\pi\rho$, and that $\rho = 0$ for uniform electric currents.

A.28.1.3 The Magnetic Fields H and $\mu\mathbf{H} = \mathbf{B}$

On p. 54 Maxwell introduced magnetic phenomena via a formal parallel with electric phenomena, which implies there are two types of magnetic fields. He labeled the magnetic "intensity" $\mathbf{H}$ as $(\alpha_1, \beta_1, \gamma_1)$ and the magnetic "induction" $\mathbf{B}$ as (a_1, b_1, c_1).[114] He called k_1 the *resistance to magnetic induction* = the reciprocal of the (relative) magnetic permeability μ, and noted (in words) that the parallel to our eq. (70), his eq. (B), is,

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \qquad [\alpha_1 = k_1 \, a_1, \ etc.], \tag{73}$$

and that in the relation $\boldsymbol{\nabla} \cdot \mathbf{B} = \mu\rho_m$, ρ_m is the density of *real magnetic matter*.[115]

A.28.1.4 Ampère's Law

On p. 56, Maxwell stated Ampère's Law in the form,[116]

$$\mathbf{J}_{\text{free}} = \boldsymbol{\nabla} \times \mathbf{H} \qquad \left[a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}, \ etc. \right], \tag{74}$$

and on p. 57 he noted that the divergence of eq. (74) is zero, so that his discussion is limited to *closed currents* that obey $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ (*i.e.*, to magnetostatics). Indeed, he added: *in fact we know little of the magnetic effects of any current that is not closed.*[117]

A.28.1.5 "Helmholtz' Theorem"

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[114]Maxwell did not use the symbol $\mathbf{B}$ for the magnetic (induction) field until 1873, in his *Treatise* [191], when he followed W. Thomson (1871), eq. (r), p. 401 of [217], who first defined $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ ($= \mu_0\mathbf{H} + \mathbf{M}$ in SI units), where $\mathbf{M}$ is the volume density of magnetization.

[115]As far as we know today, Ampère's conjecture that there is no *real magnetic matter* in Nature is correct (and magnetism is due to electric currents), so Maxwell's eq. (B) is in effect the first mathematical statement that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. As noted in sec. A.17.8 above, Faraday intuited this in 1852.

[116]This is probably the first statement of Ampère's Law as a differential equation. See sec. A.12 above for discussion of Ampère's statement of his law. See also [157].

[117]This statement can be regarded as a precursor to Maxwell's later vision (first enunciated in eq. (112), p. 19, of [169]) that all currents are closed if one considers the "displacement-current" (density) $d\mathbf{D}/dt$ in addition to the conduction-current density $\mathbf{J}_{\text{free}}$. But it also indicates that Maxwell chose not to consider the notion of moving charged particle as elements of an electrical current, as advocated by Weber (1846) [459] (see p. 88 of the English translation) as a way of understanding Ampère's expression for the force between two current loops.

For discussion of the "displacement current" of a uniformly moving charge, see [516].

There followed an interlude (pp. 57-62) on various theorems, some due to Green [184], and also Theorem VI, p. 61, which is Stokes' version, pp. 9-10 of [130], of "Helmholtz' theorem" [166, 565] that "any" vector field $\mathbf{E}$ can be related to a scalar potential Ψ and a vector potential $\mathbf{A}$ as $\mathbf{E} = -\nabla\Psi + \nabla \times \mathbf{A}$, where $\Psi = 0$ if $\nabla \cdot \mathbf{E} = 0$ and $\mathbf{A} = 0$ if $\nabla \times \mathbf{E} = 0$.[118]

A.28.1.6 Magnetic Field Energy and the Electric Field Induced by a Changing Current

After this, Maxwell considered the energy stored in the magnetic field. On p. 63, he first argued that if the magnetic field were due to a density ρ_m (ρ_1) of magnetic charges, the field could be deduced from a magnetic scalar potential Ψ_m ($= p_1$) and the energy stored in the field during the assembly of this configuration could be written,[119]

$$U_m = \frac{1}{2}\int \rho_m \Psi_m \, d\text{Vol} \qquad \left[Q = \int\int\int \rho_1 p_1 \, dxdydz \right]. \tag{75}$$

He then noted that this form can be transformed to,[120]

$$U_m = \int \frac{\mathbf{B}\cdot\mathbf{H}}{2}\, d\text{Vol} \qquad \left[Q = \frac{1}{4\pi}\int\int\int (a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1)\, dxdydz \right]. \tag{76}$$

He next argued that since this form does not include any trace of the origin of the magnetic field, it should also hold if the field is due to electrical currents, and can be transformed to,

$$U_m = \int \frac{\mathbf{J}\cdot\mathbf{A}}{2}\, d\text{Vol} \qquad \left[Q = \frac{1}{4\pi}\int\int\int \left\{ p_1\rho_1 - \frac{1}{4\pi}(\alpha_0\alpha_2 + \beta_0\beta_2 + \gamma_0\gamma_2) \right\}\, dxdydz \right]. \tag{77}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$,[121,122,123]

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[118]We also write that $\mathbf{E} = \mathbf{E}_{\text{irr}} + \mathbf{E}_{\text{rot}}$ where $\mathbf{E}_{\text{irr}} = -\nabla\Psi$ obeys $\nabla \times \mathbf{E}_{\text{irr}} = 0$ and $\mathbf{E}_{\text{rot}} = \nabla \times \mathbf{A}$ obeys $\nabla \cdot \mathbf{E}_{\text{rot}} = 0$. Many people write $\mathbf{E}_{\text{irr}} = \mathbf{E}_\parallel$ and $\mathbf{E}_{\text{rot}} = \mathbf{E}_\perp$.

[119]It seems to this author that Maxwell omitted a factor of $1/2$ in throughout his discussion on pp. 63-64.

[120]Maxwell did not discuss that the energy stored in an electric field could be written as $U_e = \int \rho_e V_e \, d\text{Vol}/2$ until Art. 85 of [190] (1973), and that this could be transformed to $U_e = \int \mathbf{E}\cdot\mathbf{D}\, d\text{Vol}/8\pi$ until Art. 631 of [191].

[121]Maxwell seems to have made a sign error in his integration by parts of the integrand $\mathbf{H}\cdot\mathbf{B} = \mathbf{H}\cdot\nabla\times\mathbf{A}$. Note that $\nabla(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{H}$.

[122]Maxwell did not seem to suppose in [158] that there is no *real magnetic matter*, so his $\mathbf{B}$ was also related to a scalar potential, and his version of eq. (77) has an additional term related to possible magnetic charges and the scalar potential.

[123]W. Thomson, p. 63 of [123] (1846), described the *electrical force* due to a unit charge at the origin *exerted at the point* (x, y, z) as $\mathbf{r}/4\pi\epsilon r^3$, without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term "electric field" were used by Thomson.

Thomson immediately continued with the example of a *small magnet*, *i.e.*, a "point" magnetic dipole $\mathbf{m}$, whose scalar potential is $\Phi = \mu\,\mathbf{m}\cdot\mathbf{r}/4\pi r^3$, noting that the *magnetic force* $(X, Y, Z) = -\nabla\Phi = \mathbf{B}$ on a unit magnetic pole p can also be written as $\nabla \times \mathbf{A}$ (although Thomson did not assign a symbol to the vector $\mathbf{A}$), where $\mathbf{A} = (\alpha, \beta, \gamma) = \mu\,\mathbf{m}\times\mathbf{r}/4\pi r^3$, with $\nabla \cdot \mathbf{A} = 0$, his eq. (2). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [116] implied this result, and is generally credited with the invention of the vector potential although the relation $\mathbf{B} = \nabla \times \mathbf{A}$ is not evident in this paper).

The time rate of change of energy in the magnetic field is $-\mathbf{J} \cdot \mathbf{E}$,[124] so, by taking the time derivative of eq. (77), and noting that $\mathbf{A} = (\alpha_0, \beta_0, \gamma_0)$ scales linearly with $\mathbf{J}$, he infers (on p. 64) that the *electro-motive force due to the action of magnets and currents is*,[125]

$$\mathbf{E}_{\text{induced}} = -\frac{\partial \mathbf{A}}{\partial t} \qquad \left[\alpha_2 = -\frac{1}{4\pi} \frac{d\alpha_0}{dt}, \ etc. \right]. \tag{78}$$

On p. 66, he stated this equation in words as: *Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction.* Here, Maxwell describes the vector potential $\mathbf{A}$ as the *electro-tonic intensity*, following Faraday, Art. 60 of [81].[126]

A.28.1.7 The Electro-Motive Force on a Moving Particle

At the bottom of p. 64, Maxwell made a statement that anticipated his later efforts towards the "Lorentz" force law: *If α_0 be expressed as a function of x, y, x, and t, and if x, y, z be the co-ordinates of a moving particle, then the electro-motive force measured in the direction of x is*,

$$\alpha_2 = -\frac{1}{4\pi} \left(\frac{d\alpha_0}{dx} \frac{dx}{dt} + \frac{d\alpha_0}{dy} \frac{dy}{dt} + \frac{d\alpha_0}{dz} \frac{dz}{dt} + \frac{d\alpha_0}{dt} \right), \qquad \mathfrak{E} = -\left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \right), \tag{79}$$

where we use the symbol $\mathfrak{E} = (\alpha_2, \beta_2, \gamma_2)$ for Maxwell's vector *electromotive force*, which is not necessarily the same as the lab-frame electric field $\mathbf{E}$. Here, Maxwell claimed that the electric field experienced by a moving particle should be computed using the convective derivative of the vector potential, and not just the partial time derivative.

If Maxwell had persisted in following the consequences of this claim, he could have deduced, via a vector-calculus identity, that the *electro-motive force* experienced by a moving particle is,

$$\mathfrak{E} = -\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) \right) = \mathbf{v} \times \mathbf{B} - \left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) \right). \tag{80}$$

If Maxwell had further considered that the electric field can have a term deducible from a scalar potential Ψ, then he might have claimed that the total *electro-motive force* on a moving particle is,

$$\mathfrak{E} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \boldsymbol{\nabla}(\Psi + \mathbf{v} \cdot \mathbf{A}) - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \tag{81}$$

We will see below (secs. A.28.2.6, A.28.3.7 and A.28.4.7) that Maxwell did not follow the path sketched above, but made important variants thereto, while others in the late 1800's

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[124]Most contemporary discussions of magnetic field energy start from the relation and work towards eq. (77) and then (76).

[125]Maxwell seems to have made another sign error, in his discussion of the time rate of change of the field energy, such that his version of our eq. (78) had the correct sign.

[126]Other mention by Faraday of the electrotonic state include Art. 1661 of [102], Arts. 1729 and 1733 of [103], and Art. 3269 of [139].

(Helmholtz [193], Larmor [214], Watson [236], J.J. Thomson in his Appendix to Chap. IX of [248], p. 260) argued that he should have proceeded as above.

A.28.1.8 Faraday's Law

Faraday's Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [158]: *the electro-motive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit,* as a summary of Faraday's comments in [137].[127] We express this as the equation (for a circuit at rest in the lab where $\mathfrak{E} = \mathbf{E}$),

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}. \tag{82}$$

A.28.2 In *On Physical Lines of Force* [167, 168, 169, 170]

A.28.2.1 The Magnetic Stress Tensor, Ampère's Law and the Biot-Savart Force Law

In [167] (1861), Maxwell considered a (linear) magnetic medium with uniform (relative) permeability μ to be analogous to a fluid filled with vortices, which led him, on p. 168, to deduce/invent the stress tensor $\mathsf{T}_{ij} = p_{ij}$ of the magnetic field,

$$\mathsf{T}_{ij} = \mu H_i H_j - \delta_{ij} 4\pi p_m \,, \tag{83}$$

where $\mathbf{H} = (\alpha, \beta, \gamma)$ is the magnetic field, and $p_m = p_1$ is a magnetic pressure. The volume force density $\mathbf{f}$ in the medium is then given by Maxwell's eq (3),

$$\mathbf{f} = \boldsymbol{\nabla} \cdot \mathsf{T}, \tag{84}$$

and the x-component of this force, $f_x = X$, is given by Maxwell's eqs. (4)-(5),

$$
\begin{aligned}
f_x &= \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \\
&= 2\mu H_x \frac{\partial H_x}{\partial x} - 4\pi \frac{\partial p_m}{\partial x} + \mu H_x \frac{\partial H_y}{\partial y} + H_y \frac{\partial H_x}{\partial y} + \mu H_x \frac{\partial H_z}{\partial z} + H_z \frac{\partial H_x}{\partial y} \\
&= H_x \boldsymbol{\nabla} \cdot \mu \mathbf{H} + \frac{\mu}{2} \frac{\partial H^2}{\partial x} - \mu H_y \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) - \mu H_z \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) - 4\pi \frac{\partial p_m}{\partial x} \\
&= H_x \boldsymbol{\nabla} \cdot \mathbf{B} + \frac{\mu}{2} \frac{\partial H^2}{\partial x} - B_y (\boldsymbol{\nabla} \times \mathbf{H})_z + B_z (\boldsymbol{\nabla} \times \mathbf{H})_y - 4\pi \frac{\partial p_m}{\partial x} \\
&= H_x \boldsymbol{\nabla} \cdot \mathbf{B} + \frac{\mu}{2} \frac{\partial H^2}{\partial x} - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{H}) - 4\pi \frac{\partial p_m}{\partial x},
\end{aligned}
\tag{85}
$$

where Maxwell introduced $\mathbf{B} = \mu \mathbf{H}$ as the magnetic induction field. In his eq. (6), p. 168, Maxwell stated that,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mu \, \rho_m \qquad \left[\frac{d}{dx} \mu\alpha + \frac{d}{dy} \mu\beta + \frac{d}{dz} \mu\gamma = 4\pi \, m \right], \tag{86}$$

---

[127]This statement appears in a letter from Maxwell to W. Thomson, Nov. 13, 1854, p. 703 of [307].

where $\rho_m = m$ is the density of *"imaginary magnetic matter"*. In his eq. (9), p. 171, Maxwell stated that,

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}} \qquad \left[\frac{1}{4\pi} \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) = p, \ etc. \right], \tag{87}$$

where $\mathbf{J}_{\text{free}} = (p, q, r)$ is the density of (free) electric current. He did not reference his discussion of our eq. (74) in his paper [158]), but arrived at eq. (87) via an argument that involved both magnetic poles and electric currents. Thus, the force density (85) can be rewritten as, Maxwell's eqs. (12)-(14), p. 172,

$$\mathbf{f} = \rho_m \mathbf{B} + \mathbf{J}_{\text{free}} \times \mathbf{B} + \boldsymbol{\nabla} \frac{\mu H^2}{2} - 4\pi \boldsymbol{\nabla} p_m. \tag{88}$$

The second term of eq. (88) is (this author believes) the first statement of what is now commonly called the Biot-Savart force law for a free electric-current density,

$$\mathbf{F} = \int \mathbf{J}_{\text{free}} \times \mathbf{B} \, d\text{Vol}, \tag{89}$$

in terms of a magnetic field (of which Biot and Savart [30] had no conception).

However, Maxwell did not consider an electric current to be a flow of charged particles, so he did not immediately interpret eq. (88) as a derivation of the "Lorentz" force $q\mathbf{v} \times \mathbf{B}$ on a moving electric charge q.

Also, Maxwell did not note at this time that the third and fourth terms of eq. (88) cancel, in that $p_m = \mu H^2/2$ is the "magnetic pressure",[128] nor did he infer that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \mu \rho_m$.

A.28.2.2 Magnetic Field Energy

On p. 63 of [158], Maxwell had deduced that the energy stored in the magnetic field can be computed according to our eq. (76) via an argument that supposed the existence of magnetic charges (monopoles) and a corresponding magnetic scalar potential. In Prop VI, pp. 286-288 of [168], Maxwell used his model of molecular vortices to deduce the same result (given in his eqs. (45)-(46), p. 288, and again in his eq. (51), p. 289),

$$U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{2} \, d\text{Vol} \qquad \left[E = \frac{1}{2} \mu (\alpha^2 + \beta^2 + \gamma^2) V \right]. \tag{90}$$

A.28.2.3 Faraday's Law

In Prop. VII, pp. 288-291 of [168], Maxwell considered the time derivative of the magnetic field energy, written in his eq. (52), p. 289, as,

$$\frac{dU_m}{dt} = \int \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \, d\text{Vol} \qquad \left[\frac{dE}{dt} = \frac{1}{4\pi} \mu V \left(\alpha \frac{d\alpha}{dt} + \beta \frac{d\beta}{dt} + \gamma \frac{d\gamma}{dt} \right) \right], \tag{91}$$

---

[128]If the permeability μ is nonuniform, the third and fourth terms combine to yield the term $(H^2/2)\boldsymbol{\nabla}\mu$, as noted by Helmholtz (1881) [205].

He also introduced the *electromotive force* $\mathfrak{E} = (P, Q, R)$ on a unit electric charge, and argued, using his model of molecular vortices, that the (electro)magnetic field does work on an electric current density $\mathbf{J}_{\text{free}}$ at rate,[129]

$$\frac{dU_m}{dt} = -\int \mathbf{J}_{\text{free}} \cdot \mathbf{E} \, d\text{Vol} \qquad \left[\frac{dE}{dt} = -\frac{1}{4\pi} \sum (Pu + Qv + Rw) dS \right]. \tag{92}$$

In cases like the present where the *electromotive force* $\mathfrak{E}$ is that on a hypothetical unit charge at rest in the lab, it is the same as the electric field $\mathbf{E}$, whose symbol will be used in this section.[130] Maxwell next gave an argument in his eqs. (48)-(50) equivalent to using our eq. (87) to find,

$$\frac{dU_m}{dt} = -\int \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) \, d\text{Vol} = -\int \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) \, d\text{Vol}. \tag{93}$$

Comparing our eqs. (91) and (93), we infer,[131] as in Maxwell's eq. (54), p. 290, that,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \left(= -\frac{\partial \mathbf{B}}{\partial t} \right) \qquad \left[\frac{dQ}{dz} - \frac{dR}{dy} = \mu \frac{d\alpha}{dt} \,, \, etc. \right]. \tag{94}$$

This is the first statement of Faraday's law as a (vector) differential equation. Surprisingly, Maxwell did not give this differential form either in [176] or in his *Treatise* [248].

A.28.2.4 $\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \ \boldsymbol{\nabla} \cdot \mathbf{A} = 0$

Also on p. 290 of [168], Maxwell discussed the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, his eq. (55), *subject to the conditions*, his eqs. (56) and (57),

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{95}$$

Maxwell gave no justification for these conditions, which are the first statements by him of them.[132] While we now recognize that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ holds in the absence of true magnetic charges, as apparently is the case in Nature (and that if $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, then $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ follows from vector calculus), the relation $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ is a choice of "gauge" (in particular, the Coulomb gauge) and not a law of Nature.

A.28.2.5 Electric Field outside a Toroidal Coil with a Time-Varying Current

We digress slightly to note that on pp. 338-339 of [168], Maxwell considered a toroidal coil in his Fig. 3. If this coil carries an electric current, there is no exterior magnetic field even in the case of a time-dependent current (if one neglects electromagnetic radiation,

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[129]Maxwell first considered his dE/dt of eq. (47), p. 289, for a surface element, extending this to a volume in his eq. (50).

[130]In sec. A.28.2.6 below, which concerns moving charges, we will use the symbol $\mathfrak{E}$ for Maxwell's *electromotive force* vector.

[131]This argument ignores a possible contribution to the field energy from the electric field.

[132]Maxwell likely followed Thomson, who considered that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in eqs. (2) and (3) of [123]. Thomson was inspired by Stokes' discussion [113] of incompressible fluid flow where the velocity vector $\mathbf{u}$ obeys $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, Stokes' eq. (13).

whose existence Maxwell reported only in the next paper, [169], in his series *On Physical Lines of Force*). However, a changing current induces an external electric field, which seems like action at a distance. Maxwell noted that while the external magnetic field is zero, the external vector potential **A** (*electrotonic state*) is not, and the external electric field is related to the time derivative of **A**. It seems that the vector potential **A** had "physical reality" for Maxwell, which view was later extended to a quantum context by Aharonov and Bohm [353, 532].

Fig. 3.
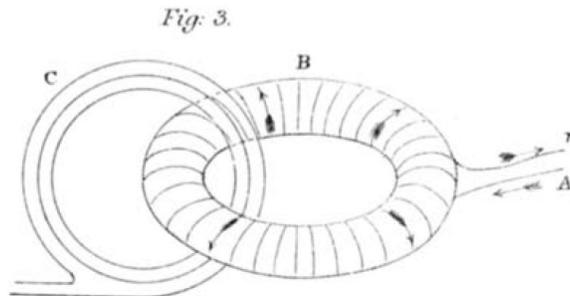
A.28.2.6 *Electromotive Force in a Moving Body*

In Prop. XI, pp. 340-341 of [168], Maxwell considered a body that might be deforming, translating, and/or rotating, and discussed the resulting changes in the magnetic field **H**. On one hand, he stated in his eq. (70), p. 341, that,[133]

$$\delta\mathbf{H} = (\delta\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{H} + \delta t\frac{\partial\mathbf{H}}{\partial t} \qquad \left[\delta\alpha = \frac{d\alpha}{dx}\delta x + \frac{d\alpha}{dy}\delta y + \frac{d\alpha}{dz}\delta z + \frac{d\alpha}{dt}\delta t\right], \tag{96}$$

which uses the convective derivative. On the other hand, he stated before his eq. (69): *The variation of the velocity of the vortices in a moving element is due to two causes—the action of the electromotive forces, and the change of form and position of the element. The whole variation of α is therefore,*[134,135]

$$\delta\mathbf{H} = -\frac{1}{\mu}\boldsymbol{\nabla}\times\mathfrak{E}\,\delta t + (\mathbf{H}\cdot\boldsymbol{\nabla})\,\delta\mathbf{x} \quad \left[\delta\alpha = \frac{1}{\mu}\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)\delta t + \alpha\frac{d}{dx}\delta x + \beta\frac{d}{dy}\delta x + \gamma\frac{d}{dz}\delta x\right].\tag{97}$$

If we accept this relation, then we can follow Maxwell that for an incompressible medium, whose velocity field obeys $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, and if $\boldsymbol{\nabla}\cdot\mathbf{H} = 0$ (which Maxwell stated to hold *in the absence of free magnetism*, then his eqs. (69)-(70) do lead to his eq. (76), p. 342,

$$\boldsymbol{\nabla}\times\left(\mathfrak{E} - \mathbf{v}\times\mathbf{B} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0$$

$$\left[\frac{d}{dz}\left(Q + \mu\gamma\frac{dx}{dt} - \mu\alpha\frac{dz}{dt} - \frac{dG}{dt}\right) - \frac{d}{dy}\left(R + \mu\alpha\frac{dy}{dt} - \mu\beta\frac{dx}{dt} - \frac{dH}{dt}\right) = 0\right]. \tag{98}$$

---

[133]I believe that our eq. (96) holds for a body that has translated by $\delta\mathbf{x}$, without rotation or deformation.

[134]In this section, which considers the *electromotive force* on a moving, unit charge, we use the symbol $\mathfrak{E}$ for this, rather than the symbol **E**.

[135]The term $(\mathbf{H}\cdot\boldsymbol{\nabla})\,\delta\mathbf{x}$ was motivated by Maxwell's Props. IX and X, pp. 340-341 of [168], but is not evident to this author. That $\partial\mathbf{H}/\partial t = -(1/\mu)\boldsymbol{\nabla}\times\mathfrak{E}$ is Faraday's law, our eq. (94), Mawvell's eq. (54).

where here Maxwell wrote the vector potential (*electrotonic components*) as $\mathbf{A} = -(F, G, H)$. This leads to Maxwell's eq. (77) for the *electromotive forces in a moving body*,

$$\mathfrak{E} = \mathbf{v} \times \mathbf{B} - \boldsymbol{\nabla}\Psi - \frac{\partial \mathbf{A}}{\partial t} \qquad \left[P = \mu\gamma\frac{dy}{dt} - \mu\beta\frac{dz}{dt} + \frac{dF}{dt} - \frac{d\Psi}{dx} , \ \ etc. \right]. \tag{99}$$

where the "function of integration" Ψ was interpreted by Maxwell as the electric scalar potential (*electric tension*).

The sense of Maxwell's derivation is that $q\mathfrak{E}$ would be the force experienced (in the lab frame) by an electric charge q in the moving body, *i.e.*,

$$\mathbf{F} = q\mathfrak{E} = q\left(\mathbf{v} \times \mathbf{B} - \boldsymbol{\nabla}\Psi - \frac{\partial \mathbf{A}}{\partial t} \right) = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \qquad (\text{"Lorentz"}), \tag{100}$$

using the relation $\mathbf{E} = -\boldsymbol{\nabla}\Psi - \partial\mathbf{A}/\partial t$ for the electric field in the lab frame. Then, eq. (99) is the first statement of the "Lorentz" force law. However, Maxwell's argument seemed to have had little impact, perhaps due to the doubtful character of his argument leading to our eq. (97).

If the body were in uniform motion with velocity $\mathbf{v}$, $\mathfrak{E}$ could be interpreted as the electric field $\mathbf{E}'$ experienced by an observer moving with the body. Then, (in Gaussian units),

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{101}$$

which is the low-velocity Lorentz transformation of the electric field $\mathbf{E}$.

A.28.2.7 Faraday's Law, Revisited

On p. 343 of [168], Maxwell considered a moving conductor, and moving circuit, in a magnetic field, with no electric field in the lab frame. In his eqs. (78)-(79) he applied his eq. (77) to a segment of a moving conductor, finding,

$$\mathfrak{E}' \cdot d\mathbf{l} = \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = -\mathbf{B} \cdot \mathbf{v} \times d\mathbf{l} = -\mathbf{B} \cdot \frac{d\mathbf{S}}{dt} \tag{102}$$

$$\left[e = S(Pl + Qm + Rn) = S\mu\alpha\left(m\frac{dz}{dt} - n\frac{dy}{dt} \right) \right],$$

where $\mathfrak{E}$ is the *electromotive force* vector with respect to the moving conductor, $d\mathbf{S}/dt = d\mathbf{x}/dt \times d\mathbf{l}$ is the area swept out by the moving line element, $d\mathbf{l}$ of the conductor in unit time.

In the case of a moving, closed circuit, the total (scalar) electromotive force is then,

$$\mathcal{E} = \oint \mathfrak{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S} = -\frac{d\Phi_m}{dt} , \tag{103}$$

*i.e., the total electromotive force in a closed conductor is measured by the change of the number of lines of force which pass through it; and this is true whether the change be produced by the motion of the conductor or by any external cause.*

Since the above argument applies only to the case of a moving circuit, it does not demonstrate Maxwell's claim that our eq. (103) also holds for a circuit at rest with the *number of*

lines of force which pass through it changing due to an *external cause, i.e.,* a time variation of the magnetic field **B**. Presumably, Maxwell meant for the reader to recall his discussion on pp. 338-339 of [168] (sec. A.28.2.5 above): *This experiment shows that, in order to produce the electromotive force, it is not necessary that the conducting wire should be placed in a field of magnetic force, or that lines of magnetic force should pass through the substance of the wire or near it. All that is required is that lines of force should pass through the circuit of the conductor, and that these lines of force should vary in quantity during the experiment.*

A.28.2.8   Electric Currents in the Model of Molecular Vortices

On p. 13 of [169], Maxwell stated: *According to our theory, the particles which form the partitions between the cells constitute the matter of electricity. The motion of these particles constitutes an electric current; the tangential force with which the particles are pressed by the matter of the cells is electromotive force, and the pressure of the particles on each other corresponds to the tension or potential of the electricity.* Similarly, on p. 86 of [170], Maxwell stated: *in this paper I have regarded magnetism as a phenomenon of rotation, and electric currents as consisting of the actual translation of particles.*
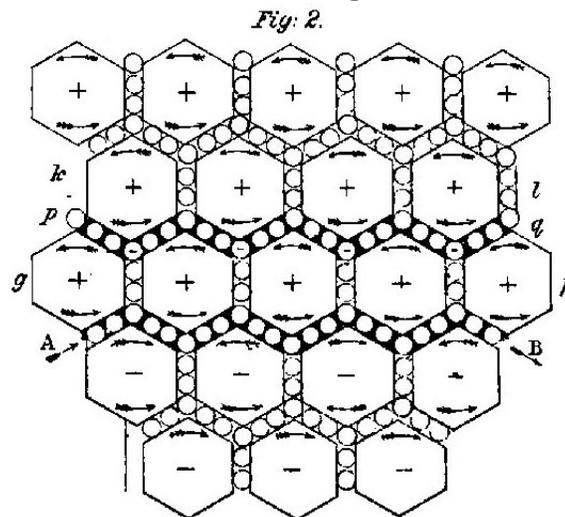
Maxwell illustrated this vision in his Fig. 2, along with the description: *Let A B, P1. V. fig. 2, represent a current of electricity in the direction from A to B.*[136]

A contemporary version of this view of electric currents in magnetic matter is that there

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[136]On p. 283 of [167], Maxwell wrote: *"What is an electric current?"*

I have found great difficulty in conceiving of the existence of vortices in a medium, side by side, revolving in the same direction about parallel axes. The contiguous portions of consecutive vortices must be moving in opposite directions; and it is difficult to understand how the motion of one part of the medium can coexist with, and even produce, an opposite motion of a part in contact with it. The only conception which has at all aided me in conceiving of this kind of motion is that of the vortices being separated by a layer of particles, revolving each on its own axis in the opposite direction to that of the vortices, so that the contiguous surfaces of the particles and of the vortices have the same motion.

In mechanism, when two wheels are intended to revolve in the same direction, a wheel is placed between them so as to be in gear with both, and this wheel is called an "idle wheel". The hypothesis about the vortices which I have to suggest is that a layer of particles, acting as idle wheels, is interposed between each vortex and the next, so that each vortex has a tendency to make the neighbouring vortices revolve in the same direction with itself.

exist a "bound" current density therein, related by,

$$\mathbf{J}_{\text{bound}} = \boldsymbol{\nabla} \times \mathbf{M}, \tag{104}$$

where $\mathbf{M}$ is the density of magnetization (*i.e.*, of Ampèrian magnetic dipoles, which are "molecular" current loops). However, this relation does not appear in *On Physical Lines of Force*, where Maxwell seemed to have supposed that his vision applied to all media, including "vacuum", and not just to magnetic matter.

A.28.2.9 Displacement Current and Electromagnetic Waves

The most novel aspect of Maxwell's paper *On Physical Lines of Force* was his introduction of the "displacement current", and his deduction that the equations of electromagnetism then imply the existence of electromagnetic waves that propagate with the speed of light.

On p. 14 of [169], his discussion reads:

Electromotive force acting on a dielectric produces a state of polarization of its parts similar in distribution to the polarity of the particles of iron under the influence of a magnet, and, like the magnetic polarization, capable of being described as a state in which every particle has its poles in opposite conditions. In a dielectric under induction, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively, and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another.

The effect of this action on the whole dielectric mass is to produce a general displacement of the electricity in a certain direction. This displacement does not amount to a current, because when it has attained a certain value it remains constant, but it is the commencement of a current, and its variations constitute currents in the positive or negative direction, according as the displacement is increasing or diminishing. The amount of the displacement depends on the nature of the body, and on the electromotive force; so that if h is the displacement (in the *z*-direction), *R the electromotive force, and E a coefficient depending on the nature of the dielectric,* $\quad R = -4\pi E^2 h$*;*
and if r is the value of the electric current (in the *z*-direction) *due to displacement,*

$$r_{\text{displacement}} = (-)\frac{dh}{dt} \;\left(= \frac{1}{4\pi E^2}\frac{dR}{dt}\right), \tag{105}$$

where it seems to this author that a minus sign should be inserted in Maxwell's original version of our eq. (105).[137]

In the above, the *electromotive force* vector $(P, Q, R) = \mathbf{E}$ is the electric field, $E^2 = 1/\epsilon$ where ϵ is the relative permittivity (dielectric constant), the *displacement* vector $(f, g, h) = -\mathbf{D}/4\pi$ is proportional to our present electric field vector $\mathbf{D}$, and $(p, q, r) = \mathbf{J}_{\text{free}}$ is the free current density. Maxwell's relation $R = -4\pi E^2 h$ (repeated in his eq. (105), p. 18, of [167]) is equivalent to,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon}\,. \tag{106}$$

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[137]For comments on reversals of signs in the relation between Maxwell's electric displacement and electric field, see [379].

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Then, Maxwell's expression for the *electric current due to displacement* is equivalent to,

$$\mathbf{J}_{\text{displacement}} = \frac{d\mathbf{D}}{dt} \,. \tag{107}$$

On p. 19 of [169], Maxwell argued that *a variation of displacement is equivalent to a current, and this current [$r_{\text{displacement}}$ of our eq. (105)] must be taken into account in equations (9) [our eq. (87)] and added to r. The three equations then become,* [Maxwell's eq. (112)],

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{d\mathbf{D}}{dt} \qquad \left[p = \frac{1}{4\pi}\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) - \frac{1}{E^2}\frac{dP}{dt} \,, \; etc. \right] . \tag{108}$$

This is the first statement of Maxwell's "fourth" equation as we know it today.

Maxwell next noted that the equation of continuity for free charge and current densities is, his eq. (113), p. 19 of [169],

$$\boldsymbol{\nabla} \cdot \mathbf{J}_{\text{free}} + \frac{\partial \rho_{\text{free}}}{\partial t} = 0 \qquad \left[\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} + \frac{de}{dt} = 0 \right] , \tag{109}$$

where $e = \rho_{\text{free}}$ is the free charge density. On taking the divergence of eq. (108) and using eq. (109), we arrive at Maxwell's eqs. (114)-(115),

$$\frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{D} = \frac{\partial \rho_{\text{free}}}{\partial t}, \qquad \boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{\text{free}} \qquad \left[e = \frac{1}{4\pi E^2}\left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \right) \right], \tag{110}$$

which is the earliest statement of Maxwell's "first" equation.[138]

In Prop. XVI, p. 22 of [169], Maxwell considered *the rate of propagation of transverse vibrations through the elastic medium of which the cells are composed,* and found the constant E in our equation (108) to have a value remarkably close to the speed of light in vacuum, and concluded *that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.*

A.28.2.10 *Electric Tension* and Poisson's Equation

A "sidelight" of Maxwell's discussion on p. 20 of [169] was his statement that in static electricity, the *electromotive force* (vector) can be related to the *electric tension* via his eq. (118), $\mathbf{E} = -\boldsymbol{\nabla}\Psi$. Further, with $\mathbf{E} = \mathbf{D}/\epsilon$, his eq. (119) (with a change of sign), and $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{\text{free}}$, his eq. (115), one has that the *tension* Ψ obeys Poisson's equation,

$$\nabla^2 \Psi = -\frac{\rho_{\text{free}}}{\epsilon} \,, \tag{111}$$

Maxwell's eq. (123) (with a change of sign).[139]

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[138]Nowadays it is more common to argue that Maxwell's "first" and "fourth" equations together imply the continuity equation (109).

[139]As will be noted in Appendices A.3.10 and A.4.3 below, the relation (111) strictly holds only in the Coulomb gauge.

A.28.3   In *A Dynamical Theory of the Electromagnetic Field* [176]

A.28.3.1   Scalar Electromotive Force and the Integral Form of Faraday's Law

In sec. 24 of [176] Maxwell considered an electrical circuit A that carries current $I_A = u$, and another circuit B that carries current $I_B = v$, and noted that the magnetic flux, Φ_m, through circuit A is given by the first equation on p. 468,

$$\Phi_m = \int_A \mathbf{B} \cdot d\mathbf{S} = LI_A + MI_B \qquad [= Lu + Mv], \tag{112}$$

where $\mathbf{B}$ is the magnetic field, $d\mathbf{S}$ is an element of the area of a surface bounded by circuit A, L is the self inductance of circuit A, and M is the mutual inductance between circuits A and B.[140] Maxwell called this flux the *momentum*, or the *reduced momentum* of the circuit.

In sec. 50, Maxwell gave a verbal statement of Faraday's Law:

1st, If any closed curve be drawn in the field, the value of M for that curve will be expressed by the number of lines of force which pass through that closed curve.

2ndly. If this curve be a conducting circuit and be moved through the field, an electromotive force will act in it, represented by the rate of decrease of the number of lines passing through the curve.

We transcribe this into symbols as,

$$\mathcal{E} = -\frac{d\Phi_m}{dt}, \tag{113}$$

where $\mathcal{E}$ is the scalar *electromotive force*.

A.28.3.2   *Electromagnetic Force* and Displacement Current

However, in sec. 56, Maxwell used the term *electromotive force* in a different way, to describe a vector, $\mathfrak{E} = (P, Q, R)$: *P represents the difference of potential per unit of length in a conductor placed in the direction of x at the given point.* This appears to mean that,

$$\mathfrak{E} = (P, Q, R) = -\boldsymbol{\nabla}\Psi, \tag{114}$$

where Ψ is Maxwell's symbol for the electric scalar potential. If so, this is the first mention of an aspect of the electric field $\mathbf{E}$ in [176], although he had introduced the *electrical displacement* $\mathbf{D} = (f, g, h)$ in sec. 55, along the with "displacement-current" (density), $(1/4\pi)\, d\mathbf{D}/dt$ in his eq. (A),

$$\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \frac{d\mathbf{D}}{dt} \qquad \left[(p', q', r) = (p, q, r) + \frac{d(f, g, h)}{dt} \right], \tag{115}$$

where the free current density $\mathbf{J}_{\text{free}} = (p, q, r)$ was introduced in sec. 54, and the *total motion of electricity* is $\mathbf{J}_{\text{total}} = (p', q', r')$. Maxwell did not use separate symbols for partial and total derivatives, so that there can be some ambiguity as to his meaning when his equations describe moving systems.

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[140]This may be the first designation of L and M as self and mutual inductances.

A.28.3.3 Vector Potential aka *Electromagnetic Momentum*

In sec. 57, Maxwell introduced the vector potential $\mathbf{A} = (F, G, H)$, but called it the *electromagnetic momentum*. In his eq. (29), Maxwell identified $-d\mathbf{A}/dt$ with *the part of the electromotive force which depends on the motion of magnets or currents.* Thus, we might now presume that Maxwell's $\mathfrak{E} = (P, Q, R)$ of his sec. 56 is the electric field,

$$\mathbf{E} = -\boldsymbol{\nabla}\Psi - \frac{\partial \mathbf{A}}{\partial t}, \tag{116}$$

but this conclusion may be premature.

A.28.3.4 Magnetic Flux aka Total *Electromagnetic Momentum of a Circuit*

In eq. (29) of sec. 58, Maxwell gave the relation for the magnetic flux Φ_m through a circuit, *the number of lines of magnetic force which pass through it,*

$$\left(\Phi_m = \int \mathbf{B} \cdot d\mathbf{S} =\right) \oint \mathbf{A} \cdot d\mathbf{l} \qquad \left[\int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds\right], \tag{117}$$

and called this the *total electromagnetic momentum* (which we must remember to distinguish from the *electromagnetic momentum* $\mathbf{A}$).

He also noted in sec. 58 that,

$$\left(\Phi_m = \oint \mathbf{A} \cdot d\mathbf{l} = \int\right) \boldsymbol{\nabla} \times \mathbf{A} \cdot d\mathbf{S} \qquad \left[\left(\frac{dH}{dy} - \frac{dG}{dz}\right) dy\, dz\right], \tag{118}$$

is the number of lines of magnetic force which pass through the area $dy\, dz$.

A.28.3.5 The Magnetic Fields H and B and Maxwell's Fourth Equation

In sec. 59, Maxwell introduced the magnetic field $\mathbf{H} = (\alpha, \beta, \gamma)$.

In sec. 60, Maxwell introduced the (relative) permeability μ, calling it the *coefficient of magnetic induction.*

In eq. (B) of sec. 61, Maxwell gave the *Equations for Magnetic Force,*

$$\mu\mathbf{H} \,(= \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{A} \qquad \left[\mu\alpha = \left(\frac{dH}{dy} - \frac{dG}{dz}\right), \ etc.\right]. \tag{119}$$

We use the symbol $\mathbf{B}$ for Maxwell's $\mu\mathbf{H}$.[141]

In eq. (C) of sec. 62, Maxwell gives a version of Ampère's Law,

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{total}} \qquad \left[\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p', \ etc.\right], \tag{120}$$

recalling from our eq. (115) that Maxwell's vector (p', q', r') is the total current density $\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{displacement}}$.

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[141]The symbol $\mathbf{B}$ for the quantity $\mu_0\mathbf{H} + \mathbf{M}$, where $\mathbf{M}$ is the magnetization density, was introduced by W. Thomson in 1871, eq. (r), p. 401 of [217], and appears in Art. 399 of Maxwell's *Treatise* [191].

A.28.3.6 *Electromotive Force in a Circuit* at Rest

In eq. (32), sec. 63 of [176], Maxwell stated that the *electromotive force acting round* an electrical circuit is related by,[142]

$$\mathcal{E} = \oint \mathfrak{E} \cdot d\mathbf{l} \qquad \left[\xi = \int \left(P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds} \right) ds \right]. \qquad (121)$$

Supposing this circuit, A, carries current $I_A = u$, and another circuit, B, carries current $I_B = v$, Maxwell, in his eq. (33), reminded us the magnetic flux through circuit A is given by our eq. (112),

$$\Phi_m = \oint_A \mathbf{A} \cdot d\mathbf{l} = LI_A + MI_B \qquad \left[\int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} \right) ds = Lu + Mv \right], \qquad (122)$$

as he had previously discussed in sec. 24. Then, his eq. (34) states that,

$$\mathcal{E}_A = -\frac{d}{dt}(LI_A + MI_B) \qquad \left[= -\int \frac{d\mathbf{A}}{dt} \cdot d\mathbf{l} \right], \qquad (123)$$

so comparison with our eq. (121) leads to the inference, Maxwell's eq. (35), that *if there is no motion of the circuit A,*

$$\mathfrak{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Psi \qquad \left[P = -\frac{dF}{dt} - \frac{d\Psi}{dx}, \ etc. \right], \qquad (124)$$

where Ψ could be any scalar function. But, the discussion in his sec. 56 led Maxwell to identify Ψ of eq. (124) as the electrical scalar potential.

In eq. (124), we have written $\partial \mathbf{A}/\partial t$, while Maxwell wrote $d\mathbf{A}/dt$, in that for an observer (at rest) of a circuit at rest, use of a convective derivative is not appropriate.

A.28.3.7 The Vector *Electromotive Force on a Moving Conductor*

In sec. 64, Maxwell deduced the force on a bar the slides on a U-shaped rail, while carrying a current, with the entire system in an external magnetic field. He gave no figure in [176], but the figure below is associated with his discussion of this example in Arts. 594-597 of [248]. C represents a battery that drives the current in the circuit.

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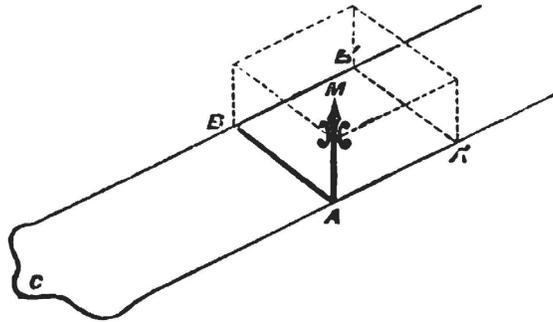
[142]This meaning of the term *electromotive force* is still in use today. However, Maxwell also used the term *electromotive force* in sec. 65 of [176] to describe the force $\mathbf{v} \times \mu\mathbf{H}$ on a moving, unit charge in a magnetic field, referring to his eq. (D).

Maxwell desired a very general discussion in sec. 64, so he considered a circuit whose plane was not perpendicular to any of the x, y or z axes, which makes his description rather intricate. Here, we suppose the circuit lies in the x-z plane, with the sliding piece, AB, of length a parallel to the x-axis, and the long arms of the U-shaped rail parallel to the z-axis, at, say $x = 0$ and a. The velocity $v_z = dz/dt$ of the sliding bar is in the z-direction, and the uniform, external magnetic field is in the y-direction.

As in sec. 63, Maxwell considered changes in the magnetic flux through the circuit, $\oint \mathbf{A} \cdot d\mathbf{l}$, our eq. (117), to infer the strength of his vector $\mathfrak{E}$. The part of the line integral over the sliding bar changes at rate,[143]

$$a \frac{dA_x}{dz} \frac{dz}{dt} , \tag{125}$$

as indicated in the first equation on p. 485. Because the length of the circuit in z is increasing, the line integral also changes at rate,

$$\frac{dz}{dt} [A_z(x=0) - A_z(x=a)] = -\frac{dz}{dt} \frac{dA_z}{dx} a, \tag{126}$$

as given in the second equation on p. 485. Hence, the total rate of change of magnetic flux, given in the third and fourth equations on p. 485, is,[144]

$$\frac{d\Phi_m}{dt} = a v_z \left(\frac{dA_x}{dz} - \frac{dA_z}{dx} \right) = a v_z B_y = -\mathcal{E} = - \oint \mathfrak{E} \cdot d\mathbf{l}. \tag{127}$$

Maxwell considered that eq. (127) describes a contribution to the *electromotive force* $\mathfrak{E}$ beyond that in eq. (124), which additional contribution would be localized to the component $\mathfrak{E}_x$ along the sliding bar (of length a), *i.e.*, $\oint \mathfrak{E} \cdot d\mathbf{l} = a\mathfrak{E}_x$. Hence, he concluded in his eq. (36) that,

$$\mathfrak{E}_x = -v_z B_y \qquad \left[P = -\mu\beta \frac{dz}{dt} \right], \qquad i.e., \qquad \mathfrak{E} = \mathbf{v} \times \mathbf{B}, \tag{128}$$

is the part of $\mathfrak{E}$ due to the motion of the sliding bar.

Finally, in sec. 65, Maxwell stated that the total *electromotive force on a moving conductor* is his eq. (D),

$$\mathfrak{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Psi \qquad \left[P = \mu \left(\gamma \frac{dy}{dt} - \beta \frac{dz}{dt} \right) - \frac{dF}{dt} - \frac{d\Psi}{dx}, \ etc. \right], \tag{129}$$

recalling his eq. (35), our eq. (124). Again, we have written $\partial \mathbf{A}/\partial t$ where Maxwell wrote $d\mathbf{A}/dt$.

Note that Maxwell's argument in sec. 65 does not address the mechanical force on the sliding bar, $I\mathbf{a} \times \mathbf{B}$, which is the subject of most present discussions of this example.

A.28.3.8 Other *General Equations of the Electromagnetic Field*

For completeness, we briefly record other general considerations by Maxwell in [176].

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[143]In eqs.(125)-(128), the quantities A_x, A_z and B_y are evaluated at the location of the sliding bar.

[144]On p. 485, *the equations of Magnetic Force (8)* should read: *the equations of Magnetic Force (B)*.

In sec. 66, Maxwell's eq. (66), $P = kf$, *etc.*, would seem to be the equivalent of $\mathbf{D} = \epsilon \mathbf{E}$, for the displacement field $\mathbf{D} = (f, g, h)$, the electric field $\mathbf{E} = (P, Q, R)$, and the (relative) permittivity $\epsilon = 1/k$.

However, in sec. 67 we read that Ohm's law can be written as $P = -\varrho p$, *etc.*, which would seem to imply that $\mathbf{E} = -\varrho \mathbf{J}$, where ϱ is the electrical resistivity ($= 1$ / conductivity) and $\mathbf{J} = (p, q, r)$ is the electric current density.[145] Then in sec. 68 we read that,

$$\boldsymbol{\nabla} \cdot \mathbf{D} = -\rho \qquad \left[e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \right], \qquad (130)$$

and in sec. 69 that the equation of continuity is,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \qquad \left[\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \right], \qquad (131)$$

where $e = \rho$ is the electric charge density. It appears that either $\mathbf{E} = -(P, Q, R)$ and $\mathbf{D} = -(f, g, h)$, or $\rho = -e$.[146]

Fortunately, in Arts. 604-619 of his Treatise [248], Maxwell presented these relations with signs that match our present usage.

In eq. (37) of sec. 71, Maxwell noted that the (magnetic) energy of a system of steady currents can be written as (the gauge-dependent form),

$$U_m = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A} \, d\text{Vol} \qquad \left[E = \frac{1}{2} \sum (Fp' + Gq' + Hr') \, dV \right], \qquad (132)$$

and that this can be transformed to (the gauge-invariant form), his eq. (38),

$$U_m = \frac{1}{8\pi} \int \mathbf{B} \cdot \mathbf{H} \, d\text{Vol} \qquad \left[E = \frac{1}{2} \sum \{ \alpha.\mu\alpha + \beta.\mu\beta + \gamma.\mu\gamma \} \, dV \right], \qquad (133)$$

He made no mention of electric energy.

A.28.3.9   *Electromagnetic Theory of Light*

Also for completeness, we include remarks on secs 91-100 of [176], where Maxwell presented his electromagnetic theory of light.

He considered plane waves in a linear, nonconducting medium with permittivity ϵ and permeability μ, such that $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$.[147] The waves propagated in direction $\hat{\mathbf{n}}$ with speed v, such that the wave fields were only functions of the single scalar $\varphi = \hat{\mathbf{n}} \cdot \mathbf{x} - vt$, as noted at the beginning of sec. 92.

For these waves, the relation between the magnetic field $\mathbf{B}$ and the vector potential $\mathbf{A}$ can be written as,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \hat{\mathbf{n}} \times \frac{\partial \mathbf{A}}{\partial \varphi}, \qquad (134)$$

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[145]There seems to be a minus sign "loose" here. In a draft of [176], p. 160 of [453], Ohm's law was written as $P = \varrho p$, *etc.*, and in Art. 609 of [248], eq. (G) reads $\mathfrak{J} = C\mathfrak{E}$, ($\mathbf{J} = \sigma \mathbf{E}$).

[146]This issue here is likely related to Maxwell's vision of electric charge as an aspect of the displacement field $\mathbf{D}$, which leads to a concept of charge density that is the negative of our present view.

[147]In this and the following section we use SI units.

Consequently, $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$, and the magnetic field vector is transverse to the direction of propagation of the wave, as noted in eq. (62), sec. 92.

In sec. 93, Maxwell used his equation $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{total}}$, noting that in the nonconducting medium the only current is the displacement current $\partial \mathbf{D}/\partial t = \epsilon \partial \mathbf{E}/\partial t$, and that the electric field can be written in terms of potentials as $\mathbf{E} = -\partial \mathbf{A}/\partial t - \boldsymbol{\nabla} \Psi$. Then, he wrote,

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{H} &= \frac{\boldsymbol{\nabla} \times \mathbf{B}}{\mu} = \frac{\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})}{\mu} = \frac{\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}}{\mu} \\
&= \mathbf{J}_{\text{total}} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = -\epsilon \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla} \Psi\right),
\end{aligned} \tag{135}
$$

which corresponds to Maxwell's eq. (68) of sec. 94. On taking the curl of this, and replacing $\boldsymbol{\nabla} \times \mathbf{A}$ by $\mathbf{B}$, he found,

$$
\nabla^2 \mathbf{B} = \epsilon \mu \frac{\partial^2 \mathbf{B}}{\partial t^2} \qquad \left[k \nabla^2 \mathbf{B} = 4\pi\mu \frac{d^2 \mathbf{B}}{dt^2}\right], \tag{136}
$$

his eq. (69). Hence, the wavespeed is $v = 1/\sqrt{\epsilon\mu}$ $[V = \sqrt{k/4\pi\mu}]$, which in air/vacuum is extremely close to the speed of light.

Maxwell did not comment on the electric field of the wave, saying (sec. 95) instead that *this wave consists entirely of magnetic disturbances.*[148,149]

A.28.3.10 Waves of the Potentials, Coulomb Gauge

In secs. 98-99 of [176], Maxwell considers waves of the potentials. He reminded the reader that his discussion in sec. 94, our eq. (135), could emphasize the potentials rather than the magnetic field, resulting in the wave equation,

$$
\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\epsilon\mu\left(\frac{\partial^2 \mathbf{A}}{\partial t^2} + \boldsymbol{\nabla}\frac{\partial \Psi}{\partial t}\right). \tag{139}
$$

Perhaps taking inspiration from sec. 82 of [128] by W. Thomson (1950), Maxwell noted that one can make a (gauge) transformation of the vector potential according to his eq. (74), p. 500,

$$
\mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\chi, \tag{140}
$$

---

[148]Maxwell could we have added an argument for the electric field by noting that,

$$
\boldsymbol{\nabla} \times \mathbf{H} = \hat{\mathbf{n}} \times \frac{\partial \mathbf{H}}{\partial \varphi} = \epsilon\frac{\partial \mathbf{E}}{\partial t} = -\epsilon v \frac{\partial \mathbf{E}}{\partial \varphi}, \quad \Rightarrow \quad \hat{\mathbf{n}} \times \mathbf{H} = \epsilon v \mathbf{E}, \quad \text{and} \quad \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{E}}{\partial \varphi} = \boldsymbol{\nabla} \cdot \mathbf{E} = 0, \tag{137}
$$

and then taking the curl of Faraday's law to find,

$$
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \mu\mathbf{H} = -\epsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{138}
$$

such that $\mathbf{E}$ is transverse to $\hat{\mathbf{n}}$ and $\mathbf{B}$, and has wave propagation at speed $v = 1/\sqrt{\epsilon\mu}$.

[149]Since Maxwell's $k/4\pi$ (our ϵ) and μ are constants determined by laboratory experiments on static systems, he could have remarked that the speed of light is the same to any (inertial) observer.

such that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}'$, and the scalar potential should be transformed to his eq. (77), where our Ψ' is Maxwell's ϕ,

$$\Psi' = \Psi + \frac{\partial \chi}{\partial t}, \tag{141}$$

such that $\mathbf{E} = -\partial \mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi = -\partial \mathbf{A}'/\partial t - \boldsymbol{\nabla}\Psi'$.

In particular, Maxwell noted that since $\boldsymbol{\nabla} \cdot \mathbf{A}' = \boldsymbol{\nabla} \cdot \mathbf{A} - \nabla^2 \chi$, we can have $\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$, his eq. (75), by taking $\nabla^2 \chi = \boldsymbol{\nabla} \cdot \mathbf{A}$, his eq. (73). With this choice of the gauge function χ, the potentials $\mathbf{A}'$ and Ψ' are in the Coulomb gauge (which Maxwell had already favored in 1861, p. 290 of [168]). The wave equation for the vector potential in the Coulomb gauge follows from eq. (139) as,

$$\nabla^2 \mathbf{A}' - \epsilon\mu\frac{\partial^2 \mathbf{A}'}{\partial t^2} - = \epsilon\mu\boldsymbol{\nabla}\frac{\partial \Psi'}{\partial t} \qquad \text{(Coulomb gauge).} \tag{142}$$

However, Maxwell stated in his eq. (78) that *we find* the right side of eq. (142) to be zero.

To see that $\Psi' = 0$ (and hence that $\boldsymbol{\nabla}\partial\Psi/\partial t = 0$) in Maxwell's example, one can transcribe Maxwell's eq. (G) of sec. 65, $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{\text{free}}$, together with eq. (E) of sec. 66 that $\mathbf{D} = \epsilon\mathbf{E}$, and eq. (35) of sec. 63 that $\mathbf{E} = -\boldsymbol{\nabla}\Psi - \partial\mathbf{A}/\partial t$, as,[150]

$$-\frac{\boldsymbol{\nabla} \cdot \mathbf{D}}{\epsilon} = -\boldsymbol{\nabla} \cdot \mathbf{E} = \nabla^2\Psi + \frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho_{\text{free}}}{\epsilon}, \tag{143}$$

such that in the Coulomb gauge, where $\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$, we have that

$$\nabla^2\Psi' = -\frac{\rho_{\text{free}}}{\epsilon} \qquad \text{(Coulomb gauge).} \tag{144}$$

For Maxwell's example of plane waves, $\rho_{\text{free}} = 0$, such that $\Psi' = \int(\rho_{\text{free}}/\epsilon r)\,d\text{Vol} = 0$. Maxwell gave the reader little clue of this lore, although one infers that he was aware of it.

In sec. 99 Maxwell argued for a stronger result, which we now report as that for plane waves, $\mathbf{E} = -\partial\mathbf{A}'/\partial t$ (and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}'$) in any gauge (where the potentials are $\mathbf{A} = \mathbf{A}' + \boldsymbol{\nabla}\chi$ and $\Psi = \Psi' - \partial\chi/\partial t$). To this author, Maxwell derivation was not convincing.[151] However, the result follows from the condition that $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ for plane waves, and Helmholtz' theorem (Appendix A.1.5 above), such that $\mathbf{E} = \mathbf{E}_{\text{rot}} = -\partial\mathbf{A}_{\text{rot}}/\partial t$, where $\boldsymbol{\nabla} \cdot \mathbf{A}_{\text{rot}} = 0$, *i.e.*, $\mathbf{A}_{\text{rot}} = \mathbf{A}' =$ the Coulomb-gauge vector potential. Then, $0 = \mathbf{E}_{\text{irr}} = -\partial\mathbf{A}_{\text{irr}}/\partial t - \boldsymbol{\nabla}\Psi$, so while in general the potentials $\mathbf{A}_{\text{irr}}$ and Ψ can be nonzero, they do not contribute to the plane wave.[152]

A.28.4   In Maxwell's *Treatise* [190, 191, 247, 248]

Maxwell's presentation in his earlier papers his novel vision of electrodynamics is perhaps superior to that in his *Treatise*.[153]

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[150]Our eq. (143) is Maxwell's eq. (79), sec. 99, except that he had Ψ' (his ϕ) in place of Ψ (and $J = \boldsymbol{\nabla} \cdot \mathbf{A}$).

[151]Last-minute corrections by Maxwell to sec. 99 of [176] are discussed in [349]. See also p. 203 of [386].

[152]In any region where $\boldsymbol{\nabla} \cdot \mathbf{E} \approx 0$, such as far from all sources of the electromagnetic fields, these fields can be deduced only from a Coulomb-gauge vector potential to a good approximation.

[153]On p. 300 of Whittaker's *History of the Theories of Aether and Electricity* [276], one reads about Maxwell: *In 1871 he returned to Cambridge as Professor of Experimental Physics; and two years later*

Appendices A.28.4.1-5 reviews comments by Maxwell leading to the difficult discussion in Arts. 598-599 of [191] on the force on a moving circuit element, reviewed here in Appendix A.28.4.7.

A.28.4.1 Articles 70-77, *On Potential Functions*

At the end of Art. 70 of [247], Maxwell wrote: *Definition of Potential. The Potential at a Point is the work which would be done on a unit of positive electricity by the electric forces if it were placed at the point without disturbing the electric distribution, and carried from that point to an infinite distance; or, what comes to the same thing, the work which must be done by an external agent in order to bring the unit of positive electricity from an infinite distance (or from any place where the potential is zero) to the given point.*

This statement is in Vol. 1 of Maxwell's *Treatise*, which deals only with static phenomenon. However, even in Vol. 2, Maxwell never acknowledged that this definition of potential is ill defined when time-dependent magnetic fields are involved, such that the work done depends on the path.

In Art. 73, *Potential due to any Electrical System*, Maxwell stated that the electric potential V can be computed from the *electric density* ρ (tacitly, of free charge) according to,

$$V = \int \frac{\rho_{\text{free}}}{4\pi\epsilon\, r}\, d\text{Vol}. \tag{145}$$

In Art. 77, *On the Equations of Laplace and Poisson*, Maxwell stated that the potential V obeys Poisson's equation,

$$\nabla^2 V = -\frac{\rho_{\text{free}}}{\epsilon}, \tag{146}$$

in present notation (Maxwell defined his symbol ∇^2 to be the negative of ours).

Equations (145)-(146) can be taken as a matter of definition in time-dependent examples, which corresponds to use of the Coulomb gauge. The wording of the *Treatise* is consistent throughout with these equations, *i.e.*, with the choice of the Coulomb gauge.[154]

A.28.4.2 Articles 489-490, *Reaction of the Magnetic System on the Electric Circuit*

---

published his Treatise on Electricity and Magnetism. *In this celebrated work is comprehended almost every branch of electric and magnetic theory; but the intention of the writer was to discuss the whole as far as possible from a single point of view, namely, that of Faraday; so that little or no account was given of the hypotheses which had been propounded in the two preceding decades by the great German electricians. So far as Maxwell's purpose was to disseminate the ideas of Faraday, it was undoubtedly fulfilled; but the* Treatise *was less successful when considered as the exposition of its author's own views. The doctrines peculiar to Maxwell—the existence of displacement-currents, and of electromagnetic vibrations identical with light were not introduced in the first volume, or in the first half of the second volume; and the account which was given of them was scarcely more complete, and was perhaps less attractive, than that which had been furnished in the original memoirs.*

[154]It may be that when Maxwell wrote in Art. 598 that Ψ represents, *according to a certain definition, the electric potential*, he had in mind the definitions of our eqs. (145)-(146) rather than that of Art. 70.

In Art. 489, Maxwell mentioned the *language of Faraday*, which gives importance to the number of lines of magnetic induction **B** which pass through a circuit.

In Art. 490, he considered the force on an element of an electric circuit that carries current i, due to a magnetic field **B**, even when the circuit is flexible. He did not derive an expression for the force, but simply stated: *We may express in the language of Quaternions both the direction and the magnitude of this force by saying that it is the vector part of the result of multiplying the vector $i\,d\mathbf{s}$, the element of the current, by the vector $\mathfrak{B}$, the magnetic induction.* That is, $d\mathbf{F} = i\,d\mathbf{s} \times \mathbf{B}$.

A.28.4.3 Articles 502-527, *Ampère's Theory*

In Arts. 502-527, Maxwell reviewed Ampère's theory of the forces on current elements, without mention of the view of Faraday (Art. 491). In Art. 525 he noted that Ampère's experiments do not lead (as claimed by Ampère [69]) to a unique expression for this force, and in Art. 526 he gave four possible forms, the first being that of Ampère, and the second he attributed to Grassmann. While we would now say that Grassmann's form is equivalent to that given by Maxwell in Art. 490, Maxwell did not make this connection. Instead, he concluded in Art. 527: *Of these four different assumptions that of Ampère is undoubtedly the best, since it is the only one with makes the forces on two elements not only equal and opposite but in the straight line which joins them.*

A.28.4.4 Articles 530-531, *General law of induction of currents*

In Art. 530 of his *Treatise* [191], Maxwell considered electromagnetic induction in four different configurations, and then stated in Art. 531:
The whole of these phenomena may be summed up in one law. When the number of lines of magnetic induction which pass through the secondary circuit in the positive direction is altered, an electromotive force acts round the circuit, which is measured by the rate of decrease of the magnetic induction through the circuit.

A.28.4.5 Article 541, *Faraday's method of stating the laws of induction with reference to the lines of magnetic force*

Here, Maxwell gave another statement of Faraday's law: *The total electromotive force acting round a circuit at any instant is measured by the rate of decrease of the number of lines of magnetic force which pass through it.*

Then, in Arts. 525-545 he reviewed views of Lenz, Helmholtz, W. Thomson and Weber on magnetic induction.

Maxwell's verbal statements of Faraday's Law in Arts. 490, 531 and 541 are consistent with our eq. (3),

$$\mathcal{EMF} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\frac{d}{dt}\int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area}, \tag{3}$$

although he never wrote this form in his *Treatise*.

A.28.4.6 Article 569, *The current is a kinetic phenomenon*

In Art. 569, Maxwell discussed his view that while electric current is a *kinetic phenomenon* caused by *Electromotive Force* (here considered to be a vector quantity), this force is not an *ordinary mechanical force*.[155] That is, Maxwell did not consider electric current to be electric charge in motion, in contrast to contemporary thinking in Germany.

However, in Art. 599, Maxwell discussed the electromotive force on a *particle* that moves in a magnetic field, as reviewed in the following section. This can be interpreted as the first statement of the Lorentz force on a moving charged particle, although this was not appreciated at the time.

A.28.4.7 Articles 598-599, *General equation of electromotive force*

In his *Treatise* [191], Maxwell argued that an *element of a circuit* (Art. 598), or a *particle* (Art. 599), which moves with velocity $\mathbf{v}$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ experiences a (vector) *electromotive force* $\mathfrak{E}$ (Art. 598), given by eq. (B) of Art. 598 and eq. (10) of Art. 599,[156]

$$\mathfrak{E} = \mathfrak{V} \times \mathfrak{B} - \dot{\mathfrak{A}} - \boldsymbol{\nabla}\Psi, \tag{147}$$

where $\mathfrak{V}$ is the velocity $\mathbf{v}$, $\mathfrak{B}$ is the magnetic field $\mathbf{B}$, $\mathfrak{A}$ is the vector potential and Ψ *represents, according to a certain definition, the electric (scalar) potential.* If we interpret *electromotive force* to mean the force per charge q of the *particle*,[157] *i.e.*, $\mathfrak{E} = \mathbf{F}/q$, then we could write eq. (147) as,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right), \tag{148}$$

noting that the electric field $\mathbf{E}$ is given (in emu) by $-\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$.[158,159,160]

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[155]*The electric current cannot be conceived except as a kinetic phenomenon. ...*

As to the velocity of the current, we have shewn that we know nothing about it, it may be the tenth of an inch in an hour, or a hundred thousand miles in a second. So far are we from knowing its absolute value in any case, that we do not even know whether what we call the positive direction is the actual direction of the motion or the reverse.

But all that we assume here is that the electric current involves motion of some kind. That which is the cause of electric currents has been called Electromotive Force. This name has long been used with great advantage, and has never led to any inconsistency in the language of science. Electromotive force is always to be understood to act on electricity only, not on the bodies in which the electricity resides. It is never to be confounded with ordinary mechanical force, which acts on bodies only, not on the electricity in them. If we ever come to know the formal relation between electricity and ordinary matter, we shall probably also know the relation between electromotive force and ordinary force.

[156]This result also appeared in eq. (77), p. 343, of [168] (1861), and in eq. (D), sec. 65, p. 485, of [176] (1864). The term *electromagentic force* was changed to *electromagnetic intensity* by J.J. Thomson in Arts. 598-601 of the 3$^{\text{rd}}$ edition of Maxwell's Treatise [248]. See also [415, 444, 472].

[157]In contrast to, for example, Weber [459], Maxwell did not present in his *Treatise* a view of an electric charge as a "particle", but rather as a state of "displaced" æther. However, in his earliest derivation of our eq. (147), his eq. (77), p. 342 of [168], Maxwell was inspired by his model of *molecular vortices* in which moving *particles* ("idler wheels") corresponded to an electric current (see also Appendix A.28.2.8 above).

For comments on Maxwell's various views on electric charge, see [390].

[158]This assumes that Maxwell's $\dot{\mathfrak{A}}$ corresponds to $\partial\mathbf{A}/\partial t$, and not to the convective derivative $D\mathbf{A}/Dt = \partial\mathbf{A}/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}$.

[159]Maxwell never used the term *electric field* as we now do, and instead spoke of the (vector) *electromotive force* or *intensity* (see Art. 44 of [190]). The distinction is important only when discussing a moving medium, as in Arts. 598-599.

[160]The relation $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$ for the electric field holds in any gauge. However, Maxwell always

Our equation (148) is now known as the Lorentz force,[161,162] and it seems seldom noted that Maxwell gave this form, perhaps because he presented eq. (10) of Art. 599 as applying to an element of a circuit rather than to a charged particle.[163] In Arts. 602-603, Maxwell discussed *the Electromotive Force acting on a Conductor which carries an Electric Current through a Magnetic field*, and clarified in his eq. (11), Art. 603 that the force on current density $\mathfrak{J}$ (**J**) is,

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \qquad [\mathfrak{F} = \mathfrak{J} \times \mathfrak{B}]. \tag{149}$$

If Maxwell had considered that a small volume of the current density is equivalent to an electric charge q times its velocity **v**, then his eq. (11), Art. 602 could also have been written as,

$$\frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \qquad [= \mathfrak{V} \times \mathfrak{B}], \tag{150}$$

which would have confirmed the interpretation we have given to our eq. (148) as the Lorentz force law. However, Maxwell ended his Chap. VIII, Part IV of his *Treatise* with Art. 603, leaving ambiguous some the meaning of that chapter.

In his Arts. 598-599, Maxwell considered a lab-frame view of a moving circuit. However, we can also interpret Maxwell's $\mathfrak{E}$ as the electric field $\mathbf{E}'$ in the frame of the moving circuit, such that Maxwell's transformation of the electric field is,[164]

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{151}$$

The transformation (151) is compatible with both magnetic Galilean relativity, eq. (155),

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worked in the Coulomb gauge, where $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, as affirmed, for example, in Art. 619. Maxwell was aware that, in the Coulomb gauge, the electric scalar potential Ψ is the instantaneous Coulomb potential, obeying Poisson's equation at any fixed time, as mentioned at the end of Art. 783. The discussion in Art. 783 is gauge invariant until the final comment about $\nabla^2 \Psi$ (in the Coulomb gauge). That is, Maxwell missed an opportunity to discuss the gauge advocated by Lorenz [180], to which he was averse [511].

[161]Lorentz actually advocated the form $\mathbf{F} = q\,(\mathbf{D} + \mathbf{v} \times \mathbf{H})$ in eq. (V), p. 21, of [257], although he seems mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [273]. That is, Lorentz considered **D** and **H**, rather than **E** and **B**, to be the microscopic electromagnetic fields.

[162]It is generally considered that Heaviside first gave the Lorentz force law (148) for electric charges in [237], but the key insight is already visible for the electric case in [221] and for the magnetic case in [227].

[163]At the end of Art. 569 Maxwell stated: *Electromotive force is always to be understood to act on electricity only, and not on the bodies in which the electricity resides. It is never to be confounded with ordinary mechanical force, which acts on bodies only, not on the electricity in them. If we ever come to know the formal relation between electricity and ordinary matter, we shall probably also know the relation between electromotive force and ordinary force.*

Thus, while Maxwell deduced the "Lorentz" force, he was not prepared to give it the physical interpretation as the electromagnetic force on a charged particle.

[164]A more direct use of Faraday's law, without invoking potentials, to deduce the electric field in the frame of a moving circuit was made in sec. 9-3, p. 160, of [360], which argument appeared earlier in sec. 86, p. 398, of [268]. An extension of this argument to deduce the full Lorentz transformation of the electromagnetic fields **E** and **B** is given in Appendix C of [535].

and the low-velocity limit of special relativity, eq. (157).[165] These two versions of relativity differ as to the transformation of the magnetic field. In particular, if $\mathbf{B} = 0$ while $\mathbf{E}$ were due to a single electric charge at rest (in the unprimed frame), then magnetic Galilean relativity predicts that the moving charge/observer would consider the magnetic field $\mathbf{B}'$ to be zero, whereas it is nonzero according to special relativity.

These themes were considered by Maxwell in Arts. 600-601, under the heading: *On the Modification of the Equations of Electromotive Force when the Axes to which they are referred are moving in Space*, which we review in Appendix A.28.4.8 below.

Details

In Art. 598, Maxwell started from the integral form of Faraday's law, that the (scalar) *electromotive force* $\mathcal{E}$ in a circuit is related to the rate of change of the magnetic flux through it by his eqs. (1)-(2),

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S} = -\frac{d}{dt}\oint \mathbf{A} \cdot d\mathbf{l} = -\oint \left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}\right) \cdot d\mathbf{l}, \qquad (158)$$

where the last form, involving the convective derivative, holds for a circuit that moves with velocity $\mathbf{v}$ with respect to the lab frame.[166] In his discussion leading to eq. (3) of Art. 598,

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[165]The notion of Galilean electrodynamics, consistent with Galilean relativity, *i.e.*, the coordinate transformation $x' = x - vt$, $y' = y$, $z' = z$, $t' = t$, seems to have been developed only in 1973 [394]. *The term Galilean relativity was first used in 1911 [278].* In Galilean electrodynamics there are no electromagnetic waves, but only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics as a whole. In contrast, electromagnetic waves can exist in the low-velocity approximation to special relativity, and, of course, propagate in vacuum with speed c.

In Galilean electrodynamics the symbol c does not represent the speed of light (as light does exist in this theory), but only the function $1/\sqrt{\epsilon_0\mu_0}$ of the (static) permittivity and permeability of the vacuum.

In fact, there are two variants of Galilean electrodynamics:

1. Electric Galilean relativity (for weak magnetic fields) in which the transformations between two inertial frames with relative velocity $\mathbf{v}$ are (sec. 2.2 of [394]),

$$\rho_e' = \rho_e, \qquad \mathbf{J}_e' = \mathbf{J}_e - \rho_e\mathbf{v}, \qquad (c\,|\rho_e| \gg |J_e|), \qquad V_e' = V_e, \qquad \mathbf{A}_e' = = \frac{\mathbf{v}}{c^2}V_e, \qquad (152)$$

$$\mathbf{E}_e' = \mathbf{E}_e, \qquad \mathbf{B}_e' = \mathbf{B}_e - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_e \qquad \mathbf{f}_e = \rho_e\mathbf{E}_e \qquad \text{(electric)}, \qquad (153)$$

where ρ and $\mathbf{J}$ are the electric charge and current densities, V and $\mathbf{A}$ are the electromagnetic scalar and vector potentials, $\mathbf{E} = -\boldsymbol{\nabla}V - \partial\mathbf{A}/\partial t$ is the electric field, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is the magnetic (induction) field,.

2. Magnetic Galilean relativity (for weak electric fields, sec. 2.3 of [394]) with transformations,

$$\rho_m' = \rho_m - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_m, \quad \mathbf{J}_m' = \mathbf{J}_m, \quad (c\,|\rho_e| \ll |J_e|), \quad V_m' = V_m - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A}_m, \quad \mathbf{A}_m' = \mathbf{A}_m, \qquad (154)$$

$$\mathbf{E}_m' = \mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m, \qquad \mathbf{B}_m' = \mathbf{B}_m \qquad \mathbf{f}_m = \rho_m\,(\mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m) \quad \text{(magnetic)}. \qquad (155)$$

For comparison, the low-velocity limit of special relativity has the transformations,

$$\rho_s' \approx \rho_s - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_s \qquad \mathbf{J}_s' \approx \mathbf{J}_s - \rho_s\mathbf{v}, \qquad V_s' \approx V_s - \mathbf{v} \cdot \mathbf{A}_s, \qquad \mathbf{A}_s' \approx \mathbf{A}_s - \frac{\mathbf{v}}{c^2}V_s, \qquad (156)$$

$$\mathbf{E}_s' \approx \mathbf{E}_s + \mathbf{v} \times \mathbf{B}_s, \qquad \mathbf{B}_s' \approx \mathbf{B}_s - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_s \qquad \text{(special relativity, } v \ll c). \qquad (157)$$

---

[166]In Maxwell's notation, $E = \mathcal{E}$, $p = \Phi_m$, $(F, G, H) = \mathbf{A}$, $(F\,dx/ds + G\,dy/ds + H\,dz/ds)\,ds = \mathbf{A} \cdot d\mathbf{l}$, $(dx/dt, dy/dt, dz/dt) = \mathbf{v}$, and $(a, b, c) = \mathbf{B}$. Note that we interpret Maxwell's $(d/dt)(F, G, H)$ as $\partial\mathbf{A}/\partial t$.

Maxwell argued for the equivalent of use of the vector-calculus identity,

$$\nabla(\mathbf{v} \cdot \mathbf{A}) = (\mathbf{v} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{v}), \tag{159}$$

which implies for the present case,

$$(\mathbf{v} \cdot \nabla)\mathbf{A} = -\mathbf{v} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{v} \cdot \mathbf{A}) = -\mathbf{v} \times \mathbf{B} + \nabla(\mathbf{v} \cdot \mathbf{A}), \tag{160}$$

$$\mathcal{E} = \oint \left(\mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} \right) \cdot d\mathbf{l} = \oint \mathfrak{E} \cdot d\mathbf{l}, \tag{161}$$

since $\oint \nabla(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{l} = 0$. Our eq. (161) corresponds to Maxwell's eqs. (4)-(5), from which we infer that the vector *electromotive force* $\mathfrak{E}$ has the form,

$$\mathfrak{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla V, \tag{162}$$

for some scalar field V (Maxwell's Ψ), that Maxwell identified with the electric scalar potential.

If it were clear that V (Ψ) is indeed the electric scalar potential, then Maxwell should be credited with having "discovered" the "Lorentz" force law. However, Helmholtz [eq. (5^d), p. 309 of [193] (1874)], Larmor [p. 12 of [214] (1884)], Watson [p. 273 of [236] (1888)), and J.J. Thomson [in his editorial note on p. 260 of [248] (1892)] argued that our eq. (160) leads to,

$$\mathcal{E} = \oint \left[\mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla(\mathbf{v} \cdot \mathbf{A}) \right] \cdot d\mathbf{l}, \tag{163}$$

so Maxwell's eq. (D) of Art. 598 and eq. (10) of Art. 599 should really be written as,

$$\mathfrak{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla(\Psi + \mathbf{v} \cdot \mathbf{A}), \tag{164}$$

where Ψ is the electric scalar potential.[167] It went unnoticed by these authors that use of eq. (164) rather than (162) would destroy the elegance of Maxwell's argument in Arts. 600-601 (discussed in Appendix A.28.4.8 below), as well as that Maxwell's earlier derivations of our eq. (162), on pp. 340-342 of [168] (discussed in Appendices A.28.2.6-7 above) and in secs. 63-65 of [176] (discussed in Appendix A.28.3.7 above). However, the practical effect of these doubts by illustrious physicists was that Maxwell has not been credited for having deduced the "Lorentz" force law, which became generally accepted only in the 1890's.

The view of this author is that Maxwell did deduce the "Lorentz" force law, although in a manner that was "not beyond a reasonable doubt".

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[167]A possible inference from eq. (164) is that the Lorentz force law should actually be,

$$\mathbf{F} = q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B} - \nabla(\mathbf{v} \cdot \mathbf{A})\right] = q\left[\mathbf{E} + (\mathbf{v} \cdot \nabla)\mathbf{A}\right] = -q\left(\nabla V + \frac{d\mathbf{A}}{dt}\right), \tag{165}$$

Some debate persists on this issue, as discussed, for example, in [480] and references therein.

Faraday's Law

Since $\oint \boldsymbol{\nabla} V \cdot d\mathbf{l} = 0$, our eq. (161) can be rewritten as,

$$
\begin{aligned}
\mathcal{E} &= \oint \left(\mathbf{v} \times \mathbf{B} - \boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} \right) \cdot d\mathbf{l} = \oint \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} + \oint \mathbf{E} \cdot d\mathbf{l} \\
&= \oint \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} - \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{Area},
\end{aligned}
\tag{166}
$$

where the integrations are based on the instantaneous location of the circuit. That is, Maxwell's eqs. (4)-(5) of Art. 598 represent, in disguised form, the decomposition of Faraday's Law, our eq. (3), into a motional $\mathcal{EMF}$ plus that induced by a changing magnetic field through a fixed loop, as in our eqs. (4) and (9) of sec. 2 above.

A.28.4.8 Articles 600-601, *The general equations referred to moving axes*

In Art. 600, Maxwell considered a moving point with respect to two coordinate systems, the lab frame where $\mathbf{x} = (x, y, z)$, and a frame moving with uniform velocity $\mathbf{v}$ respect to the lab in which the coordinates of the point are $\mathbf{x}' = (x', y', z')$, with quantities in the two frames related by Galilean transformations. Noting that a force has the same value in both frames, Maxwell deduced that the "Lorentz" force law has the same form in both frames, provided the electric scalar potential V' in the moving frame is related to lab-frame quantities by,

$$
V' = V - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A}.
\tag{167}
$$

This is the form according to the low-velocity Lorentz transformation (157), and also to the transformations of magnetic Galilean electrodynamics (155), which latter is closer in spirit to Maxwell's arguments in Arts. 600-601.

Details

In Art. 600, Maxwell consider both translations and rotations of the moving frame, but we restrict our discussion here to the case of translation only, with velocity $\mathbf{v} = (u, v, w) = (\delta x/dt, \delta y/dt, \delta z/dt)$ with respect to the lab.[168] Maxwell labeled the velocity of the moving point with respect to the moving frame by $\mathbf{u}' = d\mathbf{x}'/dt'$, while he called labeled its velocity with respect to the lab frame by $\mathbf{u} = d\mathbf{x}/dt$. Then, Maxwell stated the velocity transformation to be, eq. (1) of Art. 600,[169]

$$
\mathbf{u}' = \mathbf{u} - \mathbf{v}, \qquad i.e., \qquad \mathbf{u} = \mathbf{v} + \mathbf{u}' \qquad \left[\frac{dx}{dt} = \frac{\delta x}{dt} + \frac{dx'}{dt} \right],
\tag{168}
$$

which corresponds to the Galilean coordinate transformation,

$$
\mathbf{x}' = \mathbf{x} - \mathbf{v}t. \qquad t' = t, \qquad \boldsymbol{\nabla}' = \boldsymbol{\nabla}, \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}.
\tag{169}
$$

---

[168]For discussion of electrodynamics in a rotating frame (in which one must consider "fictitious" charges and currents, see, for example, [507].

[169]Equation (2) of Art. 600 refers to rotations of a rigid body about the origin of the moving frame.

Maxwell next considered the transformation of the time derivative of the vector potential $\mathbf{A} = (F, G, H)$ in his eq. (3), Art. 600,

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \qquad \left[\frac{dF'}{dt} = \frac{dF}{dx}\frac{\delta x}{\delta t} + \frac{dF}{dy}\frac{\delta y}{\delta t} + \frac{dF}{dz}\frac{\delta z}{\delta t} + \frac{dF}{dt} \right], \tag{170}$$

which tacitly assumed that $\mathbf{A}' = \mathbf{A}$, and hence that $\mathbf{B}' = \mathbf{B}$.[170] In eqs. (4)-(7) of Art. 600, Maxwell argued for the equivalent of use of the vector-calculus identity (159), which implies eq. (160), and hence that,

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \times \mathbf{B} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}). \tag{171}$$

Then, in eqs. (8)-(9) of Art. 600, Maxwell combined his eq. (B) of Art. 598 with our eqs. (168) and (171) to write the *electromotive force* $\mathfrak{E}$ as, in the notation of the present section,

$$\mathfrak{E} = \frac{\mathbf{u}}{c} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}V = \frac{\mathbf{u}'}{c} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}\left(V - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right). \tag{172}$$

Finally, since a force has the same value in two frames related by a Galilean transformation, Maxwell inferred that the *electromotive force* $\mathfrak{E}'$ in the moving frame can be written as.

$$\mathfrak{E}' = \mathbf{u}' \times \mathbf{B} - \frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}\left(\Psi - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A} \right) = \mathbf{u}' \times \mathbf{B}' - \frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}'\left(\Psi - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A} \right)$$
$$= \mathbf{u}' \times \mathbf{B}' - \frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}'V' = \mathbf{u}' \times \mathbf{B}' + \mathbf{E}', \tag{173}$$

where the electric scalar potential V' in the moving frame is related to lab-frame quantities by,

$$V' = V - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A} \qquad [= \Psi + \Psi']. \tag{174}$$

This is the form according to the low-velocity Lorentz transformation (157), and also to the transformations of magnetic Galilean electrodynamics (155), which latter is closer in spirit to Maxwell's arguments in Arts. 600-601.

Further, the force $\mathbf{F}'$ on a moving electric charge q in the moving frame is given by the "Lorentz" form,

$$\mathbf{F}' = q\left(\mathbf{E}' + \mathbf{u}' \times \mathbf{B}' \right), \tag{175}$$

which has the same form eq. (148) in the lab frame. As Maxwell stated at the beginning of Art. 601: *It appears from this that the electromotive force is expressed by a formula of*

---

[170]While this assumption does not correspond to the low-velocity Lorentz transformation of the field between inertial frames, it does hold for the transformation from an inertial frame to a rotating frame. Faraday considered rotating magnets in [137], and in sec. 3090, p. 31, concluded that *No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it.* That is, $\mathbf{B}' = \mathbf{B}$ relates the magnetic field in an inertial and a rotating frame. Possibly, this might have led Maxwell to infer a similar result for a moving inertial frame as well.

the same type, whether the motions of the conductors be referred to fixed axes or to axes moving in space.[171]

A.28.4.9 Articles 602-603, The "Biot-Savart" Force Law

In articles 602-603, Maxwell considered the force on a current element $I\,d\mathbf{l}$ in a circuit at rest in a magnetic field $\mathbf{B}$, and deduced the "Biot-Savart" form,[172]

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}, \qquad \mathbf{F} = \oint I\,d\mathbf{l} \times \mathbf{B}. \tag{176}$$

Some other comments on Arts. 602-603 were given around eqs. (149)-(150) above.

A.28.4.10 Article 619, *Quaternion Expressions for the Electromagnetic Equations*

Article 619 of [191] was meant as a summary of Maxwell's theory, but the transcription of material in Arts. 598-603 was awkward. This was noticed by FitzGerald [210], who attempted to improve the story, but perhaps did not succeed. FitzGerald's comments were incorporated in Art. 619 of the 3rd edition [248] of the *Treatise*, but some typos were also introduced.

Article 619 mentioned that the vector potential is subject to the condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. This is a choice, not a requirement, and corresponds to the use of the Coulomb gauge by Maxwell.[173] We have previously remarked how with this choice the scalar potential Ψ obeys the instantaneous Poisson equation $\nabla^2 \Psi = -e/\epsilon$, where e is the (free) electric-charge density, as acknowledged by Maxwell in Art. 783.

The results of Arts. 598-599, Maxwell's eq. (B) and our eq. (147), are then reproduced in Art. 619.

However, the next sentence in Art. 619 is problematic: *The equations (C) of mechanical force (Art. 603), of which the first is,*

$$X = cv - bw - e\frac{d\Psi}{dx} - m\frac{d\Omega}{dx}, \tag{177}$$

become,

$$\mathfrak{F} = \mathfrak{J} \times \mathfrak{B} - e\boldsymbol{\nabla}\Psi - m\boldsymbol{\nabla}\Omega. \tag{178}$$

---

[171]Maxwell's equations in Art. 600 do not appear to be fully consistent with this "relativistic" statement, as he noted in Art. 601. That is, his eq. (9), Art. 600, is the equivalent to $\mathfrak{E}' = \mathbf{u}' \times \mathbf{B} - \partial\mathbf{A}'/\partial t - \boldsymbol{\nabla}(\Psi + \Psi')$, where Ψ is the electric scalar potential in the lab frame, and $\Psi' = -\mathbf{v}/c^2 \cdot \mathbf{A}$ (a lab-frame quantity) according to Maxwell's eq. (6), Art. 600. Maxwell did seem to realize that in addition to expressing the *electromotive force* $\mathfrak{E}'$ in the moving frame in terms of moving-frame quantities [our eq. (173)], as well as in terms of lab-frame quantities (our eq. (147), Maxwell's eq. (10), Art. 599), he had also deduced the relation of the electric scalar potential V' in the moving frame to the lab-frame quantities $\Psi + \Psi'$, as in our eq. (174).

[172]Maxwell had argued for this in his eqs. (12)-14), p. 172, of [167] (1861), which is the first statement of the "Biot-Savart" force law in terms of a magnetic field. Biot and Savart [30] discussed the force on a magnetic "pole" due to an electric circuit, and had no concept of the magnetic field.

[173]Maxwell's preference that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ was indicated already in eq. (57), p. 290 of [168], but without justification. He expressed this preference again in Arts. 616-617, where the context is magnetostatics.

Article 603 presented only that the force density on a conduction current $\mathbf{J}_{\text{cond}}$ ($\mathfrak{K}$) is $\mathbf{f} = \mathbf{J}_{\text{cond}} \times \mathbf{B}$ ($\mathfrak{F} = \mathfrak{J} \times \mathfrak{B}$), but in Art. 619 Maxwell indicates that $\mathfrak{J} = \mathfrak{K} + \dot{\mathfrak{D}}$, ($\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{cond}} + \mathbf{J}_{\text{displacement}}$).[174] This is problematic in that there is no mechanical force on the electric-field part of displacement $\mathbf{D} = \mathbf{E} + \epsilon_0 \mathbf{P}$, where $\mathbf{P}$ is the volume density of electric dipoles (a concept not recognized by Maxwell, and only introduced much later by Lorentz).[175]

Maxwell added a term $-m\boldsymbol{\nabla}\Omega$ to his expression for the *mechanical force*. From the last sentence in Art. 619, we infer he had in mind the special case of permanent magnetism, described by a volume density $\mathfrak{J}$ ($\mathbf{M}$) of magnetic dipoles, with a corresponding volume density $m = \boldsymbol{\nabla} \cdot \mathfrak{J}$ ($= \boldsymbol{\nabla} \cdot \mathbf{M}$), such that the magnetic field $\mathfrak{H}$ ($\mathbf{H}$) $= -\boldsymbol{\nabla}\Omega$ can be deduced from a magnetic scalar potential Ω. This special case is not strictly compatible with the existence of an electric current density $\mathfrak{J}$ in the term $\mathfrak{J} \times \mathfrak{B}$.[176]

Maxwell also included a term $-e\boldsymbol{\nabla}\Psi$ in his expression for the *mechanical force*. This is clearly not the general case for electric mechanical forces on an electric charge density e, but would apply if the charge density were static, and the corresponding electric field related to a scalar potential, $\mathbf{E} = -\boldsymbol{\nabla}\Psi$. FitzGerald (1883) [210] noted that in general the electric force on charge density e is due to the total electric field $-\boldsymbol{\nabla}\Psi - \partial\mathbf{A}/\partial t$, and so suggested that $-e\boldsymbol{\nabla}\Psi$ be replaced by $e\mathfrak{E}$. This would be valid if Maxwell's $\mathfrak{E}$ represented the (lab-frame) electric field. However, the point of Arts. 598-599 was that the symbol $\mathfrak{E}$ does not represent the (lab-frame) electric field, but rather the total electric force in case of moving charge. Nonetheless, FitzGerald's suggestion was implemented in Art. 619 of the 3rd edition of Maxwell's *Treatise* [248], which has the effect that the revised expression for the *mechanical force* includes $2\mathfrak{J} \times \mathfrak{B}$.[177]

A.28.4.11 Articles 781-805, *Electromagnetic Theory of Light*

Article 783

In Art. 783 of [191], Maxwell set the stage for discussion of electromagnetic waves other than plane waves, and made a slight generalization of his discussion of the electromagnetic

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[174]In the 3rd edition [248] of the *Treatise*, the relation $\mathfrak{J} = \mathfrak{K} + \dot{\mathfrak{D}}$ was mistyped as $\mathfrak{E} = \mathfrak{K} + \mathfrak{D}$.

[175]Maxwell only regarded the relation between $\mathbf{D}$ and $\mathbf{E}$ as $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is now called the (relative) dielectric constant and/or the (relative) permittivity. See Art. 111 of [247] for Maxwell's use of the term *polarization*.

In 1885, Heaviside introduced the concept of an electret as the electrical analog of a permanent magnet [221], and proposed that the electrical analog of magnetization (density) be called electrization. He did not propose a symbol for this, nor did he write an equation such as $\mathbf{D} = \mathbf{E} + \epsilon_0 \mathbf{P}$.

The density of electric dipoles was called the polarization by Lorentz (1892) in sec. 102, p. 465 of [246], and assigned the symbol $\mathbf{M}$.

Larmor (1895), p. 738 of [256], introduced the vector (f', g', h') for what is now written as the polarization density $\mathbf{P}$, and related it to the electric field $\mathbf{E} = (P, Q, R)$ as $(f', g', h') = (K - 1)(P, Q, R)/4\pi$, *i.e.*, $\mathbf{P} = (\epsilon - 1)\mathbf{E}/4\pi = (\mathbf{D} - \mathbf{E})/4\pi$. Larmor's notation was mentioned briefly on p. 91 of [261] (1897).

The symbol $\mathbf{M}$ for dielectric polarization was changed to $\mathbf{P}$ by Lorentz on p. 263 of [265] (1902), and a relation equivalent to $\mathbf{D} = \mathbf{E} + \epsilon_0\mathbf{P}$ was given in eq. (22), p. 265. See also p. 224, and eq. (147), p. 240 of [266] (1903), which latter subsequently appeared as eq. (142), p. 155 of the textbook [268] (1904) by Abraham.

[176]The computation of mechanical forces due to magnetism is subject to ambiguities that persist in the literature to this day. For further comments by the author on this topic, see [484].

[177]This unfortunate "improvement" must have contributed to the impression in the late 1800's that Maxwell's theory of electromagnetism was hard to follow.

theory of light in secs. 91-99 of [176], by considering currents in a medium with electrical conductivity σ. Then, eqs. (1) and (2) of Art. 783 combine to give the second line of our eq. (135) becomes,

$$\mathbf{J}_{\text{total}} = \sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t} = -\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \qquad (179)$$

which generalizes the second line of our eq. (135), and our wave equation (139) becomes,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} = -\mu\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \qquad (180)$$

which is Maxwell's eq. (6) of Art. 783, noting that in the *Treatise*, his symbol ∇^2 is the negative of ours. Maxwell had also deduced this relation, but for the case of a nonconducting medium, as eq. (68), sec. 94 of [176].

In [176], Maxwell then took to the curl of eq. (180) (with $\sigma = 0$) to find a wave equation for the magnetic field, with wavespeed $1/\sqrt{\epsilon\mu}$.

In Art. 783, Maxwell took the divergence of his eq. (6), our eq. (180), to find his eq. (8), which we write as,

$$\mu\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \nabla^2\Psi\right) = 0. \qquad (181)$$

Maxwell's next sentence included the phrase: $\nabla^2\Psi$ *which is proportional to the volume-density of free electricity*, as holds in electrostatics. Here, Maxwell supposes that even in time-dependent examples, the scalar potential obeys $\nabla^2\Psi = -\rho_{\text{free}}/\epsilon_0$, as he discussed in Art. 77 for the static case. We could say that this assumption presumes use of the Coulomb gauge ($\boldsymbol{\nabla}\cdot\mathbf{A} = 0$), but in Art. 783 Maxwell appeared to deduce the Coulomb-gauge condition from his assumption. That is, he stated that in a nonconducting medium any free electricity is at rest, such that $\nabla^2\Psi$ *in independent of* t, and *hence* $J (= \boldsymbol{\nabla}\cdot\mathbf{A})$ *must be a linear function of* t, *or constant, or zero*.[178]

Maxwell concluded Art. 783 with the statement: *we may leave* $J (= \boldsymbol{\nabla}\cdot\mathbf{A})$ *and* Ψ *out of account when considering periodic disturbances*. This claim happens to be true for plane waves, as noted at the end of Appendix A.3.10 above, but is not so in general.[179]

---

[178]The case of a vector potential that is a linear function of time finds application in electrostatics, where one can set $\Psi = 0$ and $\mathbf{A} = -\mathbf{E}\,t$, which is the so-called Gibb's gauge [259, 526]. Of course, there is no wave propagation in this case.

[179]See, for example, Prob. 2 of [477], where eqs. (91)-(92) give the Coulomb-gauge potentials for an oscillating (Hertzian) electric dipole $\mathbf{p} = \mathbf{p}_0\,e^{-i\omega t}$ at the origin as,

$$\Psi = \mathbf{p}_0\cdot\hat{\mathbf{r}}\frac{e^{-i\omega t}}{4\pi\epsilon_0\,r^2} \qquad \text{(Coulomb gauge)}, \qquad (182)$$

$$\mathbf{A} = -\frac{i}{k}\mathbf{E} - \frac{i}{k}\boldsymbol{\nabla}\Psi \qquad \text{(Coulomb gauge)} \qquad (183)$$

$$= -ik\,\hat{\mathbf{r}}\times(\mathbf{p}_0\times\hat{\mathbf{r}})\frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0\,r} + [\mathbf{p}_0 - 3(\mathbf{p}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\left[\frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0\,r^2} + i\frac{e^{i(kr-\omega t)} - e^{-i\omega t}}{4\pi\epsilon_0\,kr^3}\right].$$

Note how the (periodic) vector potential (183) consists of a part that propagates with speed $\omega/k = c$, and a

In Art. 783, Maxwell appears to wish to show that waves of the vector potential also propagate with speed $1/\sqrt{\epsilon\mu}$. In a sense, eq. (180) already shows this, if we rewrite it as,

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + \epsilon\mu \frac{\partial \boldsymbol{\nabla}\Psi}{\partial t} + \sigma \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \tag{184}$$

which linear differential equation indicates that at least part of the vector potential propagates at speed $1/\sqrt{\epsilon\mu}$. But, it seems that Maxwell wished to show that there could be no other part of the vector potential that propagates with a different speed.[180]

We now know that this cannot be shown, in that one can adopt the so-called velocity gauge (see, for example, sec. 2.3.1 of [542]) in which the scalar potential Ψ propagates with any specified speed v, and the corresponding vector potential $\mathbf{A}$ has a term that propagates at speed $1/\sqrt{\epsilon\mu}$, and another of the form $\boldsymbol{\nabla}\phi$ which propagates at speed v, such that the electric field $\mathbf{E} = -\partial \mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$ and the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ propagate only with speed $1/\sqrt{\epsilon\mu}$.[181] Of course, in the velocity gauge, $\nabla^2\Psi$ is not proportional to ρ_{free}, so Maxwell's apparent assumption in Art. 783 of this relation excluded use of a velocity gauge, except for the Coulomb gauge (with $v = \infty$).

Article 784

In Art. 783, Maxwell's eq. (9) gives the wave equation for the vector potential in the Coulomb gauge (or the physically trivial variants) and in a nonconducting medium.

Article 785

Article 785 is interesting in that Maxwell considered spherical waves from a localized source, and noted that a distant observer detects wave associated with earlier behavior at the source. However, Maxwell did not relate this behavior to the retarded potentials of Lorenz [180] (1867), to which Maxwell was averse.[182]

Articles 790-791

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part that propagates "instantaneously".

The first line of eq. (183) holds for any Coulomb-gauge vector potential of angular frequency ω.

An unpublished manuscript by Maxwell from 1873 [189], probably inspired by Sellmeier [188], contained the statement: *The vibrations of molecules which have definite periods, and which produce emission and absorption of particular kinds of light, are due to forces between the parts of the molecule...* Unfortunately, Maxwell did not relate this phenomenon to oscillating electric dipoles in his electromagnetic theory.

[180]Apparently, Maxwell did not consider instantaneous action at a distance as wave propagation.

[181]The Lorenz gauge, where $\boldsymbol{\nabla} \cdot \mathbf{A} = -\epsilon\mu\,\partial\Psi/\partial t$, is the velocity gauge with $v = 1/\sqrt{\epsilon\mu}$.

[182]See, for example, [511], and Appendix A.28.5.1 below.

In sec. 95 of [176], Maxwell may have left the reader with the impression that only the magnetic field, and not the electric field, participates in waves. This possible misimpression was corrected in Arts. 790-791 of [191], which included the Fig. 66 below, illustrating the in-phase oscillations of **E** and **B** for a linearly polarized plane wave (in Gaussian units, where $E = B$ for a plane wave in vacuum).
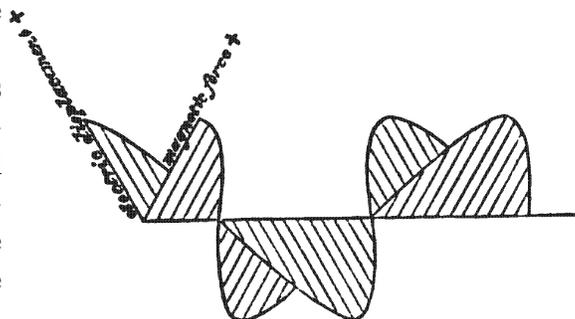
Article 798

In Art. 798, Maxwell considered a conducting medium, but (tacitly) with no free charge. Then, (in the Coulomb gauge) $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, and $\partial \Psi / \partial t = 0$, so Maxwell's eq. (6), Art. 783, our eq. (180) becomes eq. (2) of Art. 798,

$$\nabla^2 \mathbf{A} - \epsilon\mu\frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\sigma\frac{\partial \mathbf{A}}{\partial t} = 0, \tag{185}$$

Maxwell noted that this wave equation implies waves that are exponentially damped in space over a characteristic distance $1/p$, now called the skin depth.

A.28.4.12 Articles 808-831, *Magnetic Action on Light*

In these Articles, Maxwell discussed Faraday rotation (Arts. 2146-2242 of [118], 1845), which had a prominent influence on his theory of *molecular vortices*, Appendix A.28.2.8 above, as reviewed in [401].[183] Although it seems to this author that in Arts. 598-603 Maxwell was very close to supposing that massive particles could carry electric charge, he appeared in Arts. 808-831 to revert to his vision that electric charge is an aspect of an ether, separate from ordinary mass.

A.28.5   In *Note on the Electromagnetic Theory of Light* [181]

In 1868, Maxwell published a paper whose second part was titled *Note on the Electromagnetic Theory of Light* [181]. Although this paper did not bear on the issue of special relativity, we include a few remarks for completeness.[184]

A.28.5.1   Retarded Potentials

The 1868 paper is the only place where Maxwell mentioned the retarded potentials of Riemann [179] and Lorenz [180], to which he objected that they lead to violations of Newton's third law in electromagnetism (as does Maxwell's theory as well; see, for example, [302]), and also to nonconservation of energy. The latter objection was based on a misunderstanding, as reviewed by the author in [511].

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[183]A theory of Faraday rotation by C. Neumann [171], based on Weber's electrodynamics, was also reviewed in [401]. For a review of Weber's electrodynamics, see [448].

[184]Maxwell also wrote *An Elementary Treatise on Electricity*, published posthumously in 1881 [234], which considered only electro-and magnetostatics. For discussion of how this work illustrates Maxwell's thinking on electromagnetism, see [386].

A.28.5.2 Displacement Current

In 1882, FitzGerald closed his paper [208] with: *It may be worth while remarking, that no effect except light has ever yet been traced to the displacement-currents assumed by Maxwell in order to be able to assume all currents to flow in closed circuits. It has not, as far as I am aware, been ever actually demonstrated that open circuits, such as Leyden-jar discharges, produce exactly the same effects as closed circuits; and until some such effect of displacement-currents is observed, the whole theory of them will be open to question.*

Indeed, in [176, 191, 169], Maxwell discussed his concept of displacement current primarily in relation to his theory of light, so it is noteworthy that in [181] he included mention of its effect in circuits with capacitors:

Theorem D.—When the electric displacement increases or diminishes, the effect is equivalent to that of an electric current in the positive or negative direction.

Thus, if the two conductors in the last case are now joined by a wire, there will be a current in the wire from A to B.

At the same time since the electric displacement in the dielectric is diminishing, there will be an action electromagnetically equivalent to that of an electric current from B to A through the dielectric.

According to this view, the current produced in discharging a condenser is a complete circuit, and might be traced within the dielectric itself by a galvanometer properly constructed. I am not aware that this has been done, so that this part of the theory, though apparently a natural consequence of the former, has not been verified by direct experiment. The experiment would certainly be a very delicate and difficult one.

A.28.5.3 Wave Equations

Perhaps his discussion of perceived difficulties with retarded potentials sensitized Maxwell to the desirability of a deduction of a wave equation for electromagnetism that did not invoke potentials. This was provided for the magnetic field $\mathbf{H}$ in the latter part of [181].[185]

A.29 Partial Survey of Comment's on Faraday's Law since 1885

In 1885, **Heaviside** gave a verbal statement of our eq. (4) on the upper right column of p. 306 of [220]. It is generally considered that Heaviside first gave the Lorentz force law (14) for electric charges in [237], but the key insight is already visible in [221] (1885), and for hypothetical magnetic charges in [227] (1886).

In 1886, **Hoppe** [229] gave a theory of unipolar induction that was immediately contested by **Edlund**.

In 1887, **Edlund** [232] summarized his theory of unipolar induction, which included the assumption that the field lines of a rotating, cylindrical magnet rotate along with the magnet.

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[185]Maxwell considered a plane electromagnetic wave propagating in the z-direction between two infinite conducting planes perpendicular to the x-axis, with electric field $\mathbf{E} = P\,\hat{\mathbf{x}} = k\mathbf{D}$ and magnetic field $\mathbf{H} = \beta\,\hat{\mathbf{y}} = \mathbf{B}/\mu$, where $k = 1/\epsilon_0$ and $\mu = \mu_0$. He wrote the wavespeed $V = c$ as $V^2 = k/4\pi\mu$ in his mixed system of units, which becomes $c^2 = 1/\epsilon_0\mu_0$ in SI units.

In 1890, **Hertz** [242] published a paper on the fundamental equations of electromagnetics for bodies in motion, where his eq. (1_a) corresponded to $\boldsymbol{\nabla} \times \mathbf{E} = \partial \mathbf{B}/\partial t + \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B})$. However, in his sec. 4 he did not arrive at any very useful result when applying this to induction in a moving circuit.[186,187]

In 1890, **Poincaré** discussed the electromotive force induced in one circuit by changes in either the shape of, or current I in, another circuit, p. 144 of [243],
$\mathcal{E} = d(MI)/dt = M\, dI/dt + I\, dM/dt$, where M is the mutual inductance of the two circuits. Then on pp. 145-148 he transcribed Arts. 598-599 of Maxwell's *Treatise* [191], but without crisp connection of this to our eqs. (7) and (9) (although he did agree that Maxwell's Ψ is the electric scalar potential).

In 1891 and 1893, **Boltzmann** [244, 249] published his lectures on Maxwell's theory of electricity and light, and an English translation of these was published in 1987 by **Curry** [260]. Chapter 28, sec. 42 of [260] follows Arts. 598-599 of Maxwell's *Treatise* [248], with eq. (23), p. 433 corresponding to the generalized flux rule (3), noting that the electromotive force vector (P, Q, R) corresponds to our $\mathbf{E}+\mathbf{v}\times\mathbf{B}$, and recalling the statement on p. 426 that: *It follows from this form of our equations that the magnetic and electric forces are diminished in magnitude by the quantities $(\alpha_1, \beta_1, \gamma_1)$ and (P_1, Q_1, R_1) respectively. (P_1, Q_1, R_1) are the electric forces due to the motion of the given body through the magnetic field; we could thus designate them as the electromotive forces induced by motion.* That is, Boltzmann understood the equivalence of our eqs. (3) and (9) (and may have been the first after Maxwell to do so), although his statement of this was (like Maxwell's) not very crisp.

Lorentz (1892), p. 405 of [246], still identified the last form of eq. (5) as the *force électromotrice*, and wrote the "flux law" in eq. (42), p. 416, as
$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -(d/dt) \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area}$. And, in 1903, eq. 27, p. 83 of [267], Lorentz again referred to this equation, but with the proviso that the loop be at rest. Lorentz is associated with the force law (14), although he actually advocated the form $\mathbf{F} = q\,(\mathbf{D} + \mathbf{v} \times \mathbf{H})$ in eq. (V), p. 21, of [257] (1895), where he seemed mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [273].

In 1894, **Chrystal** and **Shaw**, p. 78 of [252], appeared to analyze Faraday's disk dynamo using the generalized flux law (3) for a deforming circuit, but it is not clear they were successful.

In 1894, **Föppl**'s textbook presented Faraday's law, our eq. (1), in sec. 69, p. 171 with little connection to experiment. Then, in part 5, pp. 307-355 on the electrodynamics of moving systems he made little use of Faraday's law, instead emphasizing the motional $\mathcal{EMF}$, sec. 118, pp. 320-321.

In 1900, **Poincaré** [264] gave a discussion of unipolar induction (with no figures!), asking the reader to consider a deforming circuit like that shown on the right of p. 5 above. He then computed the $\mathcal{EMF}$ from the time variation of the magnetic flux through that circuit. This may be the first analysis of Faraday's disk dynamo using the generalized flux rule (3) rather than considering the motional $\mathcal{EMF}$. Poincaré himself seemed mainly concerned with the classic issue of whether the magnetic field lines of a cylindrically symmetric magnet rotate

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[186]Hertz' doctoral thesis (1880) [200] was on induction in rotating spheres, a problem previously studied by Jochmann (1864) [175]. Related papers from this era include [202, 196, 201, 207, 214, 216, 223, 224, 225, 231, 233, 235, 241, 255].

[187]For additional commentary, see [372, 449].

when the magnet rotates, arguing that this question doesn't make sense (*i.e.*, the lines of a static field configuration don't rotate).

The equivalence of our eqs. (3) and (9) was discussed in the influential (German) text of **Abraham** (1904) [268], sec. 86, p. 398. However, his analysis of unipolar induction, sec. 88, p. 409, was based only on motional $\mathcal{EMF}$.

In 1905, **Einstein** [269] argued that from the perspective of his new theory of relativity, *questions as to the "seat" of electrodynamic electromotive forces (unipolar machines) now have no point.* See also sec. 3.3.1 above.

Starting in 1908, **Hering** argued [271, 272, 290, 292] that ambiguities in the analyses of moving circuits implied that the laws of electrodynamics needed modification. An extensive debate on Hering's views appeared in vol. **75** of *The Electrician* (1915) [284].[188]

Steinmetz (1908) argued in a comment on Hering's claims, p. 1352 of [272], that the "flux rule" should be applied only to fixed loops, and that the Biot-Savart/Lorentz force law should be considered in cases of moving circuits/conductors. See also [275].

In 1911, **Corbino** [277] reported an experiment with an annular disk in a magnetic field, where the inner and outer radii of the disk were held at different electric potentials. This configuration leads to a magnetoresistance, previously considered by **Boltzmann** [226]. See also [502]

Debates on unipolar induction, and on the issue of rotation or not of the field lines of a rotating magnet, continued throughout the 1910's, including papers by **Barnett** [279, 281], **Kennard** [280, 287], **Wilson** and **Wilson** [282], **Blondel** [283, 284],[189] **Howe** [286], **Pegram** [288]. Continued debates in the 1920's include **Swann** [289] and **Tate** [291], which latter is a useful review.

The first detailed discussion in English of the equivalence of our eqs. (3) and (9) may be that by **Bewley** (1929) in Appendix I of [294]. Textbook discussions of this in English include that by **Becker** (1932), pp. 139-142 of [297], by **Sommerfeld** (1952), pp. 286-288 of [345], by **Panofsky** and **Phillips** (1959), pp. 160-163 of [360], and by **Zangwill** (2012), sec. 14.4 of [525]. Of these, only [360] used the generalized flux rule (3) when discussing Faraday's disk dynamo.

Dwight (1930) [295] discussed how either our eq. (3) or (9) can be used to analyze examples like that considered in Appendix A.28.3.7 above, but did not comment on Faraday's disk dynamo. This paper includes interesting commentary by **Slepian** and by **Bewley**.

Page and **Adams** (1935), in a paper titled *Some Common Misconceptions in the Theory of Electricity* [302], stated on p. 56 that while most students are taught Faraday's law as our eq. (3), it actually should be given by our eq. (9).

Cramp and **Norgrove** (1935) [305, 306] reported repetitions of various experiments of Faraday meant to illustrate the view that the field lines of a rotating, cylindrically symmetric magnet do not rotate. They added a comment that if the lines did rotate, their velocity at points far from the axis of the magnet would exceed the speed of light. The paper [305] was followed by an interesting discussion with **Cooper** and **Howe**.

In 1939, **Cullwick** [310] reported a flux-linkage experiment that he felt violated the

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[188]See sec.2.4.2 above. Other comments by the author on Hering's arguments are in [551].

[189]For discussion by the author of an interesting flux-linkage experiment of Blondel, see [515].

generalized flux rule (3), and required use of our eq. (9) to analyze. See sec. 2.4.5 above.[190]

In 1945, **Sohon** published an amusing puzzler [314] that can be resolved by careful use of the generalized flux rule (3), as well as by eq. (9).

In 1945, **Tripp** published an article [315] which suggested that electrical engineers have little awareness of the force law for charged particles, as they are more concerned with macroscopic charge and current distributions. This led to a long series of articles and letters, many of which emphasized aspects of Faraday's law.[191] A comment by **Sard** [317] (1947) led to a comment by **Cullwick** [318] that mentioned a puzzle in electromagnetic induction, which led to a comment by **Slepian** [319].

In 1946, Sears [316] discussed Faraday's disk dynamo in sec. 12-5, p. 290, of his textbook, considering it to be an example of motional $\mathcal{EMF}$ rather than of Faraday's flux rule.

Perhaps inspired by the preceding dialogue, **Slepian** published a lengthy series of "engineering" puzzlers in 1947-1951. The first of these to be related to Faraday's law was [322, 325], as reviewed in sec. 2.4.7 above. In [328] he gave an amusing illustration of why Maxwell's definition of the magnetic flux through a circuit as $\Phi = \int \mathbf{B} \cdot d\mathbf{Area}$ is superior to Faraday's vision of the number of *magnetic curves* linked by the circuit.[192] Other puzzlers by Slepian related to Faraday's law and electromagnetic induction include [329] which affirmed (as first intuited by Faraday) that the relation $\mathbf{E} = -\boldsymbol{\nabla} V - \partial \mathbf{A}/\partial t$ does not imply there are two types of electric field [330],[193] which reaffirmed that magnetic flux is defined only with respect to a two-sided surface bounded by a closed curve, and six essays on motionally induced electric fields [333]-[340].

In 1949, **Cohn** [324, 331, 342] published an attempt to clarify the relation between the generalized flux law (3) and the motional $\mathcal{EMF}$ (6), but was only partially successful. Comments by **Savage** [326] and by **Bewley** [332, 343] added some clarification, but perhaps left an overall impression of confusion on this issue.

In 1952, **Corson** [344] discussed the equivalence of our eqs. (3) and (9), and used Faraday's disk dynamo as an example.

In 1954, **Hammond** [347] argued for the equivalence of our eqs. (5) and (9), but his conclusion in sec. (16), p. 155 are somewhat confusing (to the present author). This paper was followed by an extensive discussion.

In 1954, **Carrothers** and **Kapp** [348] extended Hammond's discussion [347] of Faraday's disk dynamo and Hering's examples (sec. 2.4.2 above).

In 1960 and 1962, **Then** [355, 358] discussed variants of Faraday's disk dynamo as examples of motional $\mathcal{EMF}$.

In 1961, **Birss** [356] discussed the equivalence of our eqs. (3) and (9), using Faraday's disk dynamo as an example, and then appeared to argue that the magnetic field rotates with a rotating, cylindrical magnet.

In 1961, **Strickler** [357] discussed Faraday's disk dynamo via motional $\mathcal{EMF}$, arguing

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[190]Cullwick elaborated upon his doubts in Chap. 18 of his text [354].

[191]Tripp appears to have been an advocate of Weber's action-at-a-distance electrodynamics (see Appendix A.23 above), noting that in this the interaction of two charge particles obeys Newton's third law, whereas the Lorentz force law (14) does not. Further comments by Tripp on this theme can be found in [313, 320].

[192]Other comments by Slepian on subtleties of the notion of field lines are in [321, 341].

[193]In comments on this puzzler [333, 334], Slepian gave a definition of $\mathcal{EMF}$ based on some concept of an electrostatic field, which does not stand scrutiny in this author's view.

that the rotating disk is the seat of the $\mathcal{EMF}$.

In 1963, **Sears** [361] discussed the generalized flux law (5), but only for circuits at rest.[194]

In 1963, **Das Gupta** [362] discussed unipolar generators, emphasizing that the magnetic field does not rotate along with a cylindrically symmetric magnet.

In 1963, **Webster** [364] discussed Faraday's disk dynamo in the rotating frame of the disk, using general covariance. As noted in sec. 3.3 above, this complicated analysis does not add much to understanding in the inertial lab frame.

In 1963, **Pugh** [367] discussed the equivalence of our eqs. (3) and (9), as well as Hering's 1908 circuit example [272] (also sec. 2.4.2 above). He also argued that when considering physical $\mathcal{EMF}$s that are associated with work done on an electric charge as it traverses a circuit, it is helpful to consider the flow of energy as described by the Poynting vector $\mathbf{S} = \mu_0 \mathbf{E} \times \mathbf{B}$.

In 1963, **Feymnan** discussed Faraday's disk dynamo in sec. 17-2 of [369], emphasizing our eq. (9) rather than the generalized flux rule (3). He also gave a second example in his Fig. 17-3 of an exception to the fixed-loop flux law, our eq. (2); however, the (small) $\mathcal{EMF}$ in this example is equally well computed by either our eq. (3) or (9) (using the "obvious" current path).

In 1964, **Bewley** [370] updated his 1952 monograph on the generalized flux law. This is perhaps the most extensive published discussion of complicated examples of induced $\mathcal{EMF}$s. Bewley claimed to show: *how ridiculously easy it is to understand and apply Faraday's* (generalized flux) *law to any and all cases*, but the present author finds much of his discussion to be extremely intricate.

In 1964, **Bhagwat** and **Simmonds** [371] argued in favor of the decomposition (9) instead of the generalized flux law (3).

In 1965, **Woodward** [375] discussed the equivalence of our eqs. (5) and (9), including some comments on the views of "engineers" and "physicists".

In 1967, **Kaempffer** [380], and also **Tilley** [382], gave variants of Hering's examples of circuits that change in a manner that violates the generalized flux rule (3), but is correctly explained by eq. (9). See secs. 2.4.2-2 above.

In 1967, **Carter** published the second edition of his "engineering" textbook [376], which included "Carter's Rule", an interesting qualification to the generalized flux rule (3) (see sec. 2.3 above, and [556]). This important statement has not been mentioned elsewhere that I can find.

In 1967, **Laithwaite** [377] published an interesting commentary on *Electromagnetic Puzzles*, including several variants of Faraday's disk dynamo.

In 1968, **Rosser** [381] mentioned Faraday's disk dynamo on p. 164 of his book, claiming that it could be explained via motional $\mathcal{EMF}$, but not via Faraday's flux rule. See also [388].

In 1969, **Scanlon**, **Henriksen** and **Allen** [384] discussed the equivalence of our eqs. (3) and (9) via a (to this author) distracting use of relativity and moving frames, although these equations both apply to the lab frame in which the observer (galvanometer) is at rest. They discussed Faraday's disk dynamo, Tilley's example [382], and Kaempffer's example [380] and

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[194]On p. 440 it was claimed that no magnetic field exists outside a toroidal winding even when the current is time varying. This is a misunderstanding. See, for example, eq. (22) of [508].

acknowledged Bewley's approach [370] to these, while preferring a different rationalization for the validity of the generalized flux rule in these cases.

In 1972, **Nussbaum** [392] discussed the equivalence of our eqs. (3) and (9), and argued that the flux law should be applied only to cases where work is done. That is, he rejected the abstract formulation of our eq. (2) for closed curves not associated with a physical conductor.

In 1977, **Bartlett**, **Monroy** and **Reeves** [403] reported a measurement that set a very strong limit against possible rotation of the magnetic field of a cylindrical, rotating magnet.

In 1978, **Crooks** *et al.* [406] discussed Faraday's disk dynamo, and the equivalence of our eqs. (3) and (9), in a manner marred (in the view of the present author) by claims about the physics in the rotating frame of the disk. See also [408].

In 1981, **Miller** [416] published a lengthy survey of unipolar induction, emphasizing the theory of relativity.

In 1982, **Rosen** and **Schieber** [417] discussed how many attempts to relate Faraday's law to the theory of relativity have gone awry.

In 1983, **Marazzini** and **Tucci** [419] discussed the influence of Faraday on Einstein.

In 1987, **Müller** published the first of several controversial discussions [429, 521] on Faraday's disk dynamo.

In 1989. **Ares de Parga** and **Rosales** [433] published an article claiming to be the first discussion of the relation between Faraday's law and the theory of relativity.

In 1989, **Lorrain** [435] gave an interesting discussion of electric-charge distributions in Faraday's dynamo based on a conducting sphere.

In 1990, **Gelman** [440] gave yet another discussion of the relation between Faraday's law and the theory of relativity.

In 1990, **Marinov** posed a puzzler [439] based on Faraday's law, discussed in [550].

From 1993 to 1995, an editor of IEE News, writing as **"Dipole"**, challenged his readers with several variants of the Faraday disk dynamo, and the issue of whether the magnetic fields rotates with a rotating, cylindrical magnet [443]. No statements of facts or opinions were offered.

In 1994, **Assis** and **Thober** [447] published a historical review of Weber's approach to unipolar induction (which term Weber coined in 1839 [107]).

In 1994, **Guala-Valverde** and **Mazzoni** [450] noted that for there to be motional $\mathcal{EMF}$ there must be (relative) motion. Their later commentaries include [478, 481, 482]. See also comments by a sometime collaborator, **Achilles** [487].

In 1996, **Rajaraman** [456] discussed an example that appeared in Fig. 2 of [295], and also in Fig. 31, p. 69, of [370], which concerns the electric field on a conductor at rest in the fields of two moving magnets. Rather than working in the lab frame, Rajaraman recommended working in the two moving frames of the two magnets, in which the conductor is moving and the concept of motional $\mathcal{EMF}$ can be used. However, Rajaraman then implied that the lab-frame analysis is "wrong" (although it gave the correct result). *This is another example of the attitude about multiple approaches to a problem which led to the present note.*

In 1996, **Galili** and **Kaplan** [457] argued that the "flux rule" (3) is of "limited validity", and appeared to favor use of eq. (9) which they characterized as a "microscopic" approach.

In 1998-2002, **Kelly** [465, 468, 490] repeated Faraday's dynamo experiment with a rotating, cylindrical magnet, and claimed this "proved" that the magnetic field lines rotate with the magnet.

In 1998, **Layton** and **Simon** [461] discussed the subtleties of several experiments on motional $\mathcal{EMF}$.

In 1999, **Carpenter** [464] (and in many previous papers) advocated analysis of electromagnetic induction using the Lorenz-gauge vector potential, claiming the Maxwell favored this (although the latter always used the Coulomb-gauge vector potential).

In 1999, **Montgomery** [466] discussed the "neglected topic" of unipolar induction, claiming the equivalence of our eqs. (3) and (9), but only using the latter in his analysis. He published additional comments on this topic in [491].

In 2000, **Hertzberg** *et al.* [473] reported a repeat of the Wilson-Wilson experiment [282], in agreement with the standard view that physics in the inertial lab frame can be related to that at a point in a rotating object via a Lorentz transformation to a locally comoving frame of reference (while electrodynamics in the rotating frame is more complicated).

In 2002, **Cavicchi** [485] discussed "productive confusions" in learning about electromagnetic induction.

In 2003, **Munley** [492] discussed challenges to the "flux rule" without noticing that this rule means different things to different people. He advocated use of the generalized flux (3), but his concluding sec. V contains several erroneous statements.

In 2005, **Berg** and **Alley** [494] posted a claim that the unipolar generator is *perhaps the only table-top demonstration of special relativity*.

In 2005, **Galili**, **Kaplan** and **Mehavi** [496] discussed the equivalence of our eq. (3) and (9), and then repeated the typical distraction that the lab-frame Lorentz force (14) can/should be related to the electric field according to a moving observer, although there is no such observer in any known example of Faraday's disk dynamo.

In 2008, **Giuliani** [502] argued that our eq. (9) is the basic method for computation of $\mathcal{EMF}$s, and its equivalence to eq. (3) provides an occasionally useful computational shortcut. He is one of the few to note that Maxwell's Arts. 598-599 of [191] discussed these things. However, in 2010 he claimed [513] that some examples of electromagnetic induction cannot be explained without use of a vector potential, which therefore must be "real".

In 2008, **López-Ramos**, **Menéndez** and **Piqué** [505] discussed the equivalence of our eqs. (3) and (9), emphasizing that changes in the shape of the circuit must be continuous for the generalized flux law (3) to apply. The experiments of Hering [272] and Blondel [284] were mentioned (but not cited).

In 2011, **Leus** and **Taylor** [518] reported variants of experiments with disk dynamos involving rotating, asymmetric magnetic fields. They seemed to imply that such experiments "prove" that the magnetic field of a rotating, cylindrically symmetric magnetic rotates with the magnet.

In 2012, **Rajaraman** [520] claimed that the magnetic field of a rotating, cylindrically symmetric magnet rotates with the magnet.

In 2012, **Macleod** [522] argued that the magnetic field of a rotating, cylindrically symmetric magnet does not rotate with the magnet.

In 2012, **Nezhad** [524] wrote his PhD thesis on homopolar generators.

In 2016, **Chen** *et al.* reported an experiment on unipolar induction which was interpreted as showing that the magnetic field does not rotate with a cylindrical magnet.

In 2017, **Bordoni** [546] wrote a review emphasizing nonstandard interpretations of electromagnetic induction.

In 2017, **Bilbao** [547] discussed the equivalence of our eqs. (5) and (9), but seemed to imply that in case of an extended conductor the analysis is independent of the path of integration.

In 2019, **Zengel** [552] advocated use of the generalized flux law (3), which he claimed was not due to Maxwell, following the mis-statement about this in [271].

In 2019, **Matveev** and **Matvjev** [554] gave a discussion of unipolar induction emphasizing motional $\mathcal{EMF}$.

In 2019, **Härtel** [557] argued that in view of the ambiguities of the description of Faraday's disk dynamo in Maxwellian electrodynamics, one should instead use Weber's electrodynamics.

In 2021, **Giuliani** [559] continued his arguments about the "flux rule", now claiming that it is "not a physical law", and is inconsistent with special relativity. See comments on this by the present author in [567].

In 2024, **Davis** [568] lost track of the distinction between $d\Phi_{\mathbf{B}}/dt$ and $\partial\Phi_{\mathbf{B}}/\partial t$ in his eq. (28).

The ambiguities in the use of Faraday's law in examples of electromagnetic induction, based on the concept of electromagnetic fields, lead some people to seek analyses based only on electric currents, without mention of fields. This can, of course, be done in Maxwell's electrodynamics, but it is also a feature of Weber's electrodynamics. As such, there exists a small community that favors Weber's views (which do not include electromagnetic radiation) over those of Faraday and Maxwell. See, for example, [564] and references therein.

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